CHARACTERIZATION OF REGULARITY VIA VARIATIONAL STABILITY OF ALTERNATING PROJECTIONS SEQUENCES

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ABSTRACT. The notion of regular pair (A, B) for two nonempty closed convex subsets A and B of a Hilbert space $\mathcal{H}$ was introduced by Borwein and Bauschke in 1993 to ensure convergence (in norm) of the alternating projection method to some point of the best approximation set. In 2022, De Bernardi and Miglierina showed that regularity of the pair (A, B) guarantees, additionally, the convergence for any variational perturbation of the alternating projection method, provided the corresponding best approximation sets are bounded. The aim of this paper is to show that the converse assertion is also true.

Keywords: Alternating projection method, Convex feasibility problem, d-stability, regularity, Attouch-Wets convergence.

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CONTENTS

1. INTRODUCTION

Let A and B be two nonempty closed convex sets in a Hilbert space $\mathcal{H}$. The 2-set convex feasibility problem consists of finding a pair of points $a \in A$ and $b \in B$ realizing the distance between A and B, that is, $d(A, B) = \|a - b\|$. Clearly, a necessary condition for the existence of such a pair is that the *best approximation sets*

$$E := \{a \in A;\, \mathrm{dist}(a, B) = \mathrm{dist}(A, B)\}, \qquad F := \{b \in B;\, \mathrm{dist}(b, A) = \mathrm{dist}(A, B)\} \quad (1.1)$$

are nonempty. In this work we shall assume that this condition is always satisfied and consequently the problem is well-posed. Notice that this happens whenever the intersection of the sets A and B is nonempty (in this case, $E = F = A \cap B$).

The method of alternating projections is the simplest iterative procedure for finding a solution of the convex feasibility problem and it goes back to von Neumann [11]: let us denote by P_A and P_B the projections onto the sets A and B, respectively. Then for any starting point $a_0 \in \mathcal{H}$, consider the *alternating projection sequences* $\{a_n\}_{n\geq 1} \subset A$ and $\{b_n\}_{n\geqslant 1} \subset B$ defined inductively by

$$b_n = P_B(a_{n-1}) \ \text{ and } \ a_n = P_A(b_n) \qquad (n \in \mathbb{N}). \tag{1.2}$$

If the sequences $\{a_n\}$ and $\{b_n\}$ converge in norm, then we say that the method of alternating projections (also known as the von Neumann method) converges.

It is well-known that the sequences $\{a_n\}$ and $\{b_n\}$ are always weakly convergent ([5]). Notwithstanding, in the separable Hilbert space ℓ_2, a celebrated example of Hundal in 2004 ([9]) shows that in general the alternating projections method may fail to converge in norm. This was the starting point for investigations and several conditions ensuring the convergence of the von Neumann method appeared in the literature.

Besides the case in which A and B are linear subspaces (this was the original setting studied by von Neumann himself), one of the best known and most important sufficient conditions for the convergence of such a method is the notion of *regularity* for the pair (A, B) (see forthcoming Definition 2.6), introduced by Bauschke and Borwein in [2]. This notion guarantees the norm convergence of the sequences defined in (1.2).

In the more recent papers [6–8], the authors studied stability properties of the alternating projection method. More precisely, for any two sequences of nonempty closed convex sets $\{A_n\}_{n\geqslant 1}$ and $\{B_n\}_{n\geqslant 1}$ such that $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$ for the Attouch-Wets variational convergence (see forthcoming Definition 2.2) and any initial point $a_0 \in \mathcal{H}$, they considered the *perturbed alternating projection sequences* $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, defined as follows:

$$b_n = P_{B_n}(a_{n-1}) \quad \text{and} \quad a_n = P_{A_n}(b_n) \qquad (n \in \mathbb{N}). \tag{1.3}$$

In particular, in [6], they showed that the aforementioned regularity condition is not only sufficient to guarantee the convergence in norm of the alternating projection sequence in (1.2), but also of its *variational* version given by (1.3). To be more precise, using the notation of (1.1) let us recall the notion of d-stability introduced in [6].

Definition 1.1 (d-stability)**.** We say that the pair (A, B) of two nonempty closed convex subsets of a Hilbert space $\mathcal{H}$ is *d-stable*, whenever all perturbed alternating projection sequences $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ given in (1.3) satisfy

$$\lim_{n\to\infty} \operatorname{dist}(a_n, E) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \operatorname{dist}(b_n, F) = 0 \qquad \text{(in norm)}.$$

The main result of [6] asserts that if the pair (A, B) is regular and the sets E and F are bounded, then the pair (A, B) is d-stable. In the same paper, the authors asked whether the inverse implication holds, stating the following open problem:

Problem 1.2. *Is every d-stable pair (A, B) necessarily regular?*

The main aim of the present paper is to show that Problem 1.2 has a positive answer under the condition that the best approximation set E is bounded (equivalently, F is bounded). In particular, we establish the following result.

Theorem A. *Let A, B be nonempty closed convex subsets of $\mathcal{H}$ for which the best approximation sets E and F are nonempty and bounded. Then the following are equivalent:*

(i) *the pair (A, B) is regular;*

(ii) *every variational alternating projection algorithm converges (that is, the pair (A, B) is d-stable).*

In this work, we establish implication (ii) $\Longrightarrow$ (i) (see forthcoming Theorem 3.7). The other implication was previously established in [6, Theorem 4.9]. Alltogether, Theorem A provides a complete characterization of regularity by means of variational stability of alternating projection sequences.

We resume this introduction with a brief description of the structure of the paper. Section 2 is devoted to fix our notation and review preliminary notions. Definitions and properties of the Hausdorff and respectively, the Attouch Wets set convergence are therein recalled, together with relevant properties of metric projections and of alternating projections method in a Hilbert space. In particular, the relationship between regularity and d-stability obtained in [6] is recalled. In Section 3, we prove the main result of the paper, namely, the fact that d-stability implies regularity for a given convex feasibility problem. For the convenience of the reader, we first prove the main result under a stronger assumption (namely, that the involved sets A and B are separated by a linear functional and A is bounded). Then, we eventually remove the extra assumptions and establish the result in the general case.

2. Notation and preliminaries

Throughout this paper, we will always work with a Hilbert space $\mathcal{H}$ equipped with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. We denote by $\mathbb{B}_{\mathcal{H}}$ and $\mathbb{S}_{\mathcal{H}}$ the closed unit ball and the unit sphere of $\mathcal{H}$, respectively. If $\alpha > 0$, $x \in X$, and $A, B \subset \mathcal{H}$, we denote as usual

$$x + A := \{x + a;\, a \in A\}, \quad \alpha A := \{\alpha a;\, a \in A\}, \quad A + B := \{a + b;\, a \in A,\, b \in B\}.$$

For any two points $x, y \in \mathcal{H}$, we denote by $[x, y]$ the closed segment in $\mathcal{H}$ with endpoints x and y and we set $(x, y) = [x, y] \setminus \{x, y\}$ for the corresponding "open" segment. For a subset A of $\mathcal{H}$, we denote by $\operatorname{int}(A)$, $\operatorname{conv}(A)$ and $\overline{\operatorname{conv}}(A)$ the interior, the convex hull and the closed convex hull of A, respectively. We further define the distance of a point $x \in X$ to a set A as follows:

$$\operatorname{dist}(x, A) := \inf_{a \in A} \|a - x\|.$$

We are going to use the following *simplified notation*. If f is a real-valued function defined on $\mathcal{H}$, we denote

$$[f \leqslant \alpha] := \{x \in \mathcal{H};\, f(x) \leqslant \alpha\} \quad \text{for } \alpha \in \mathbb{R}.$$

The sets $[f \geqslant \alpha]$ and $[f = \alpha]$ are defined similarly.

2.1. **Hausdorff and Attouch-Wets convergence of sequences of sets.** Let us now review two notions of convergence for a sequence of sets (for a more detailed overview of this topic, see, e.g., [3]). By $\mathfrak{c}(\mathcal{H})$ we denote the family of all nonempty closed subsets of $\mathcal{H}$. Let us introduce the (extended) Hausdorff metric $\mathrm{D_H}$ in $\mathfrak{c}(\mathcal{H})$. For $A, B \in \mathfrak{c}(\mathcal{H})$, we define the excess of A over B as follows:

$$e(A,B) := \sup_{a\in A} \mathrm{dist}(a,B).$$

If $A \neq \emptyset$ and $B = \emptyset$ we set $e(A,B) = \infty$, while if $A = \emptyset$ we set $e(A,B) = 0$. Furthermore, we define the Hausdorff distance between sets A and B as follows:

$$\mathrm{D_H}(A,B) := \max\big\{\, e(A,B),\, e(B,A)\,\big\}.$$

The above metric gives rise to our first notion of set convergence.

Definition 2.1 (Hausdorff convergence)**.** We say that a sequence $\{A_n\}$ in $\mathfrak{c}(\mathcal{H})$ converges *in the Hausdorff sense* to a set $A \in \mathfrak{c}(\mathcal{H})$ if

$$\lim_{n\to\infty} \mathrm{D_H}(A_n, A) = 0\,.$$

According to [10, Theorem 8.2.12] the sequence $\{A_n\}_{n\geqslant 1}$ in $\mathfrak{c}(\mathcal{H})$ is Hausdorff converging to a set $A \in \mathfrak{c}(\mathcal{H})$ if and only if

$$\sup_{x\in X} |\,\mathrm{dist}(x,A_n) - \mathrm{dist}(x,A)| \underset{n\to\infty}{\longrightarrow} 0\,.$$

We now consider a weaker notion of convergence, the so-called Attouch-Wets convergence (see, e.g., [10, Definition 8.2.13]). To this end, fix $r > 0$, $A, B \in \mathfrak{c}(\mathcal{H})$ and set:

$$\begin{aligned} e_r(A,B) \;&:=\; e(A\cap r\,\mathbb{B}_{\mathcal{H}},\, B) \;\in [0,\infty),\\ \mathrm{D}_{\mathrm{H},r}(A,B) \;&:=\; \max\{e_r(A,B),\, e_r(B,A)\}. \end{aligned}$$

The above family of pseudo-distances gives rise to the following notion of convergence:

Definition 2.2 (Attouch-Wets convergence)**.** We say that a sequence $\{A_n\}_{n\geqslant 1}$ in $\mathfrak{c}(\mathcal{H})$ converges in the *Attouch-Wets sense* to a set $A \in \mathfrak{c}(\mathcal{H})$ if for every $r > 0$ we have:

$$\lim_{n\to\infty} \mathrm{D}_{\mathrm{H},\,r}(A_n, A) = 0\,.$$

The Attouch-Wets convergence can be seen as a "localization" of the Hausdorff convergence. Indeed, according to [10, Theorem 8.2.14], a sequence $\{A_n\}_{n\geqslant 1}$ in $\mathfrak{c}(\mathcal{H})$ is Attouch-Wets converging to a set $A \in \mathfrak{c}(\mathcal{H})$ if and only if for every $r > 0$

$$\sup_{x\in r\mathbb{B}_{\mathcal{H}}} |\,\mathrm{dist}(x,A_j) - \mathrm{dist}(x,A)| \underset{n\to\infty}{\longrightarrow} 0\,.$$

The following property relates Hausdorff and Attouch-Wets convergences.

Fact 2.3. *Let $A \in \mathfrak{c}(\mathcal{H})$, $\{A_n\}_{n\geqslant 1} \subset \mathfrak{c}(\mathcal{H})$, $\{r_n\}_n \subset (0,+\infty)$ increasingly converging to $+\infty$, and $\{\delta_n\}_n \subset (0,+\infty)$ converging to 0. Assume that $\mathrm{D_H}(A\cap r_n\mathbb{B}_{\mathcal{H}}, A_n) \leqslant \delta_n$, whenever $n \geqslant 1$. Then, for every $r > 0$, the inequality $\mathrm{D}_{\mathrm{H},r}(A, A_n) \leqslant \delta_n$ eventually holds as $n \to \infty$. In particular, the sequence $\{A_n\}_{n\geqslant 1}$ is Attouch-Wets converging to A.*

Proof. Let $r \in \mathbb{N}$. For every $n \in \mathbb{N}$, we clearly have

$$e_r(A_n, A) = e(A_n \cap r\mathbb{B}_{\mathcal{H}}, A) \leqslant e(A_n, A \cap r_n\mathbb{B}_{\mathcal{H}}) \leqslant \delta_n.$$

Moreover, if $n_0 \in \mathbb{N}$ is such that $r_{n_0} > r$, then for all $n \geqslant n_0$ we have

$$e_r(A, A_n) = e(A \cap r\mathbb{B}_{\mathcal{H}}, A_n) \leqslant e(A \cap r_n\mathbb{B}_{\mathcal{H}}, A_n) \leqslant \delta_n\,.$$

We have shown that $\mathrm{D}_{\mathrm{H},r}(A, A_n) \leqslant \delta_n$ eventually holds as $n \to \infty$. The final conclusion, about the Attouch-Wets convergence of $\{A_n\}_{n\geqslant 1}$, immediately follows. ■

2.2. **Projections in Hilbert spaces and the alternating projections method.** The notions of distance between two convex sets and of projection of a point onto a convex set of a Hilbert space play a fundamental role in our paper. The projection onto a nonempty closed convex subset C maps any point $x_0 \in \mathcal{H}$ to its nearest point in C, denoted by $P_C(x_0)$. We shall frequently use in the paper the following result, usually called *variational characterization of the projection onto* C. Let $c_0 \in C$ and $x_0 \in \mathcal{H}$. Then $c_0 = P_C(x_0)$ if and only if

$$\langle x_0 - c_0, c - c_0\rangle \leqslant 0, \quad \text{for all} \;\; c \in C. \tag{2.1}$$

We recall that the projection P_C is a nonexpansive map from $\mathcal{H}$ to C, i.e., it holds $\|P_C(x) - P_C(y)\| \leqslant \|x - y\|$ (see, e.g., [**10**, Proposition 10.4.8]).

In the sequel, we shall need the following elementary fact, the proof of which is left to the reader.

Fact 2.4. *Let $C, D \subset \mathcal{H}$ be nonempty closed convex sets such that $C \subset D$. The following assertions hold.*

(a) If C is an affine set then P_C is an affine function.
(b) If $b, p \in \mathcal{H}$ are such that $p = P_D(b)$ and $p \in C$, then $p = P_C(b)$.

Given $n \in \mathbb{N}$ and two convex sets $A, B \in \mathfrak{c}(\mathcal{H})$, let us consider the nonexpansive map $\Pi_n^{A,B} : \mathcal{H} \to \mathcal{H}$, defined by

$$\Pi_0^{A,B}(x) = x, \quad \Pi_n^{A,B}(x) = \underbrace{P_A P_B \ldots P_A P_B}_{n \text{ times}}(x), \qquad x \in \mathcal{H}.$$

Given A and B in $\mathfrak{c}(\mathcal{H})$, let $\{a_n\}_n$, $\{b_n\}_n$ be the corresponding alternating projection sequences as in (1.2), starting from the initial point $x \in \mathcal{H}$, then we can rewrite them in terms of the above operator $\Pi_n^{A,B}(\,\cdot\,)$ as follows:

$$a_n = \Pi_n^{A,B}(x)\,, \qquad b_n = P_B(\Pi_n^{A,B}(x)).$$

Let us now consider two nonempty closed convex subsets A and B and define the *displacement vector*

$$v := P_{\overline{B-A}}(0) \qquad \text{for the pair } \;(A, B)\,. \tag{2.2}$$

It is clear that if $A \cap B \neq \emptyset$, then $E = F = A \cap B$ and the displacement vector for the pair (A, B) is null. We denote by $\mathrm{Fix}(T)$ the set of all fixed points of an operator $T : \mathcal{H} \to \mathcal{H}$. Recalling from (1.1) the definition of the best approximation sets E and F we have:

Fact 2.5 ([2, Fact 1.1])**.** *Suppose that $\mathcal{H}$ is a Hilbert space and that A, B are nonempty closed convex subsets of $\mathcal{H}$. Then:*

(i) $\|v\| = \operatorname{dist}(A, B)$ *and* $E + v = F$*;*

(ii) $E = \operatorname{Fix}(P_A P_B) = A \cap (B - v)$ *and* $F = \operatorname{Fix}(P_B P_A) = B \cap (A + v)$*;*

(iii) *For every $e \in E$ and $f \in F$ we have:*

$$P_B e = P_F e = e + v \qquad \text{and} \qquad P_A f = P_E f = f - v.$$

2.3. **Regularity and d-stability for a pair of convex sets.** We shall now state two notions of regularity for a pair of nonempty closed convex sets A and B, which have been introduced in [2] to ensure convergence in norm for the alternating projection algorithm (see also [4]).

Definition 2.6 (regular and boundedly regular pair)**.** Let A and B be two nonempty closed convex subsets of a Hilbert space $\mathcal{H}$, such that the corresponding best approximation sets E and F are nonempty.

The pair (A, B) is called:

(i) *regular* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in \mathcal{H}$ we have:

$$\max\{\operatorname{dist}(x, A), \operatorname{dist}(x, B - v)\} \leqslant \delta \quad \Longrightarrow \quad \operatorname{dist}(x, E) \leqslant \varepsilon\,. \tag{2.3}$$

(ii) *boundedly regular*, if for every bounded set $S \subset \mathcal{H}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that (2.3) holds for every $x \in S$.

Bounded regularity condition is always fulfilled in finite-dimensions (see [2, Theorem 3.9]). We refer the reader to [2] for concrete examples of pair of sets satisfying (or failing to satisfy) properties (i) and (ii).

The following statement shows that bounded regularity is equivalent to regularity whenever the best approximation sets E, F are bounded, see [6, Prop. 3.2].

Proposition 2.7. *Let A, B be nonempty closed convex subsets of a Hilbert space $\mathcal{H}$. If the pair (A, B) is regular, then it is also boundedly regular (and E, F are nonempty). Moreover, if E is bounded, the converse also holds.*

The next result follows from [2, Theorem 3.7] and [6, Theorem 4.9]. Indeed, recalling Definition 1.1 we have:

Theorem 2.8. *Let A, B be nonempty closed convex subsets of a Hilbert space $\mathcal{H}$ so that the pair (A, B) is regular. The following assertions hold:*

(i) *The alternating projection method converges in norm;*

(ii) *If E, F are bounded, the pair (A, B) is d-stable.*

3. Main Results

In this section, our aim is to prove Theorem A. To this end, we are going to make use of several intermediate results and technical lemmas, which we are going to obtain progressively. Let us start with the following important intermediate result that provides a framework under which a sequence generated by the alternating projection method remains in a 2-dimensional space.

Lemma 3.1 (keeping iterations in a 2-dim space)**.** *Let z, w be two nonzero elements of a Hilbert space $\mathcal{H}$ such that $\langle w, z\rangle = 0$ and define functions:*

$$x \mapsto f(x) = \langle z, x\rangle \quad \text{and} \quad x \mapsto \widehat{f}(x) = \langle z + \alpha w, x\rangle\,, \quad \text{for } \alpha := \frac{\|z\|^2}{\|w\|^2}. \tag{3.1}$$

Let further A, B be nonempty closed convex subsets of $\mathcal{H}$ such that

$$[z, w] \subseteq A \subseteq \left[\widehat{f} \geqslant \widehat{f}(w)\right] \qquad \text{and} \qquad [0, w] \subseteq B \subseteq [f \leqslant 0]. \tag{3.2}$$

Then the alternating projection method starting from $a_0 := P_A(0)$ generates a sequence that converges to w, that is:

$$\lim_{n\to\infty} \Pi_n^{A,B}\left(P_A(0)\right) = \lim_{n\to\infty} \Pi_n^{B,A}\left(0\right) = w \in A \cap B\,.$$

Proof. We claim that:

(i) if $a \in [z, w]$ then $P_B(a) \in [0, w]$;

(ii) if $b \in [0, w]$ then $P_A(b) \in [z, w]$.

To prove (i), observe that $P_{[f\leqslant 0]}(a) = P_{[f=0]}(a)$. Since $P_{[f=0]}(z) = 0$ and $P_{[f=0]}(w) = w$, by Fact 2.4(a), we have that $P_{[f=0]}(a) \in [0, w]$. The proof of (i) follows by our assumptions and by Fact 2.4(b).

To prove (ii), observe that $P_{[\widehat{f}=\widehat{f}(w)]}(-\alpha w) = z$ and $P_{[\widehat{f}=\widehat{f}(w)]}(w) = w$. Arguing as in the proof of (i), we have that if $b \in [-\alpha w, w]$ then $P_A(b) \in [z, w]$. In particular, (ii) holds and the claim is proved.

Therefore, setting

$$\widetilde{A} := [z, w] \qquad \text{and} \qquad \widetilde{B} := [0, w].$$

we deduce that

$$\Pi_n^{A,B}\left(P_A(0)\right) = \Pi_n^{\widetilde{A},\widetilde{B}}\left(P_A(0)\right)\,.$$

Since $\Pi_n^{\widetilde{A},\widetilde{B}}\left(P_A(0)\right) \to w$ as $n \to \infty$ (see, e.g., [2, Fact 1.2]), the proof is complete. ■

We shall now consider two nonempty closed convex sets A and B that are separated by a functional and show that, provided one of them is bounded, we can perturb the sets to obtain a setting in which the previous lemma can be applied.

Lemma 3.2. *Let A, B be nonempty closed convex subsets of $\mathcal{H}$ such that for some $f \in S_{\mathcal{H}^*}$, $\beta \in \mathbb{R}$ and $r \geqslant 1$ we have:*

$$\sup f(B) \leqslant \beta \leqslant \inf f(A) \qquad \text{and} \qquad A \subset r\mathbb{B}_{\mathcal{H}}\,.$$

Let further $p, w \in [f = \beta]$, $p \neq w$ and suppose that for some $\delta > 0$

$$(p + \delta\mathbb{B}_{\mathcal{H}}) \cap A \neq \emptyset, \quad (p + \delta\mathbb{B}_{\mathcal{H}}) \cap B \neq \emptyset, \quad (w + \delta\mathbb{B}_{\mathcal{H}}) \cap A \neq \emptyset, \quad (w + \delta\mathbb{B}_{\mathcal{H}}) \cap B \neq \emptyset\,.$$

Then there exist closed convex sets $\widehat{A}, \widehat{B}$ *such that*

$$\mathrm{D_H}(A, \widehat{A}) \leqslant 3\delta \quad \textit{and} \quad \mathrm{D_H}(B, \widehat{B}) \leqslant 3\delta \tag{3.3}$$

so that the corresponding alternating projection sequence satisfies:

$$\left\{\Pi_n^{\widehat{B},\widehat{A}}(p)\right\}_{n\geqslant 1} \subset [f = \beta] \qquad \textit{and} \qquad \lim_{n\to\infty} \Pi_n^{\widehat{B},\widehat{A}}(p) = w.$$

Proof. Without any loss of generality, we may assume that $p = 0$ and consequently $\beta = 0$. Define

$$\widehat{B} = (B + 3\delta\mathbb{B}_{\mathcal{H}}) \cap [f \leqslant 0].$$

Since $\mathrm{D_H}(B, B + 3\delta\mathbb{B}_{\mathcal{H}}) \leqslant 3\delta$ and $B \subseteq [f \leqslant 0]$, we have $B \subset \widehat{B} \subset B + 3\delta\mathbb{B}_{\mathcal{H}}$ and hence

$$\mathrm{D_H}(B, \widehat{B}) \leqslant 3\delta.$$

Set $\varepsilon_0 = \min\{1, \|w\|\}$, let u be an element in $\mathbb{S}_{\mathcal{H}}$ representing the functional $f \in \mathbb{S}_{\mathcal{H}^*}$ (that is, $f(\cdot) = \langle u, \cdot\rangle$), and define:

$$\widehat{f}(x) = \langle u + \theta w, x\rangle = f(x) + \theta\langle w, x\rangle, \quad x \in \mathcal{H}, \qquad \text{where} \quad \theta = \frac{\delta\varepsilon_0}{r\|w\|^2}. \tag{3.4}$$

Since $\|u\| = 1$ and $\langle u, w\rangle = 0$, we have $\|\widehat{f}\| > 1$. Moreover, for every $a \in A$ we have:

$$\widehat{f}(a) = f(a) + \theta\langle w, a\rangle \geqslant -\theta\|w\|\,\|a\|. \tag{3.5}$$

We now consider the set

$$\widehat{A} = (A + 3\delta\mathbb{B}_{\mathcal{H}}) \bigcap \left[\widehat{f} \geqslant \widehat{f}(w)\right].$$

Claim. $\mathrm{D_H}(A, \widehat{A}) \leqslant 3\delta$.

Proof of the claim. Since $\widehat{A} \subset A + 3\delta\mathbb{B}_{\mathcal{H}}$, we have $e(\widehat{A}, A) \leqslant 3\delta$. Let now $a \in A$. If $a \in \left[\widehat{f} \geqslant \widehat{f}(w)\right]$, then $a \in \widehat{A}$, while if $\widehat{f}(a) < \widehat{f}(w)$, we set $\widetilde{a} = P_{\left[\widehat{f}\geqslant\widehat{f}(w)\right]}(a) = P_{\left[\widehat{f}=\widehat{f}(w)\right]}(a)$. Then $\widehat{f}(\widetilde{a}) = \widehat{f}(w) = \delta\varepsilon_0 r^{-1}$ and we obtain from (3.5) that

$$\|\widetilde{a} - a\| = \frac{\left|\widehat{f}(w) - \widehat{f}(a)\right|}{\|\widehat{f}\|} \leqslant \widehat{f}(w) - \widehat{f}(a) \leqslant 2\delta.$$

It follows that $\widetilde{a} \in \widehat{A}$, whence $e(A, \widehat{A}) \leqslant 2\delta$ and the claim follows.

Finally, consider the triangle with vertices $0, w$ and $z := \frac{\delta\varepsilon_0}{r}u$. By the definition of the sets $\widehat{A}$ and $\widehat{B}$, we deduce that

$$[0, w] \subset \widehat{B} \qquad \text{and} \qquad \left[\frac{\delta\varepsilon_0}{r}u, w\right] \subset \widehat{A}.$$

Therefore, we can apply Lemma 3.1 for the sets $\widehat{A}, \widehat{B}$ and for

$$z = \frac{\delta\varepsilon_0}{r}u, \qquad \text{and} \qquad \alpha = \frac{\|z\|^2}{\|w\|^2}$$

to conclude that

$$\lim_{n\to\infty} \Pi_n^{\widehat{A},\widehat{B}}\left(P_{\widehat{A}}(0)\right) = \lim_{n\to\infty} \Pi_n^{\widehat{B},\widehat{A}}(0) = w.$$

The proof is complete. ■

We are now ready to establish our main result. For the reader´s convenience, we shall first do so under the additional assumption that the sets A, B are separated by a functional $f \in \mathcal{H}^*$. This assumption, which will eventually be dropped in the sequel, simplifies considerably the proof, allowing us to outline the geometrical features of the arguments.

We recall from (1.1) and (2.2) the definitions of the best approximation sets E, F and the displacement vector for a pair (A, B) of nonempty closed convex sets in $\mathcal{H}$ and start by proving the following technical lemma.

Lemma 3.3. *Let (A, B) be a pair of nonempty closed convex sets in $\mathcal{H}$ such that*

$$\sup f(B) \leqslant 0 = \inf f(A) \qquad \textit{for some } f \in \mathbb{S}_{\mathcal{H}^*}.$$

Assume further that E, F are nonempty, bounded and the pair (A, B) is not regular. Then, there exist $\varepsilon_0 > 0$, $r > 0$ and sequences

$$\{\delta_n\}_{n\geqslant 1} \subset (0, +\infty) \textit{ with } \lim_{n\to\infty} \delta_n = 0 \qquad \textit{and} \qquad \{w_n\}_{n\geqslant 1} \subset \ker f \,\cap\, (E + r\,\mathbb{B}_{\mathcal{H}})$$

such that:

$$\mathrm{dist}(w_n, E) > \varepsilon_0 \qquad \textit{and} \qquad \max\Big\{\mathrm{dist}(w_n, A),\, \mathrm{dist}(w_n, B - v)\Big\} \leqslant \delta_n\,.$$

Proof. We shall show the result for the case $A \cap B \neq \emptyset$. The general case follows by an easy adaptation of the same arguments.

Since the pair (A, B) is not regular, there exist $\{w_n'\}_{n\geqslant 1} \subset \mathcal{H}$ and $\varepsilon_0 > 0$ such that

$$\mathrm{dist}(w_n', A \cap B) > 3\varepsilon_0, \quad \text{for all } n \geqslant 1 \quad \text{and} \quad \lim_{n\to\infty} \max\left\{\mathrm{dist}(w_n', A),\, \mathrm{dist}(w_n', B)\right\} = 0.$$

Then setting $w_n'' := P_{\ker f}(w_n')$, one has

$$\|w_n' - w_n''\| \leqslant \max\{\mathrm{dist}(w_n', A), \mathrm{dist}(w_n', B)\},$$

yielding

$$\lim_{n\to\infty} \|w_n' - w_n''\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \max\left\{\mathrm{dist}(w_n'', A),\, \mathrm{dist}(w_n'', B)\right\} = 0.$$

Then for some $n_0 \in \mathbb{N}$ sufficiently large and all $n \geq n_0$ we have:

$$\mathrm{dist}(w_n'', A \cap B) \geqslant \mathrm{dist}(w_n', A \cap B) - \|w_n' - w_n''\| > 2\varepsilon_0\,.$$

We now pick $w_n \in [w_n'', P_{A\cap B}(w_n'')]$ so that $\varepsilon_0' < \mathrm{dist}(w_n, A \cap B) < \varepsilon_0' + 1$. Therefore, since $A \cap B$ is bounded, there exists $r > 0$ such that $w_n \in (A \cap B) + r\mathbb{B}_{\mathcal{H}}$. Finally, by convexity of the sets A and B we have

$$\delta_n := \max\{\mathrm{dist}(w_n, A), \mathrm{dist}(w_n, B)\} \leqslant \max\left\{\mathrm{dist}(w_n'', A),\, \mathrm{dist}(w_n'', B)\right\}$$

and the proof is complete. ■

We are now ready to establish the result under the additional assumption that the sets are separated by a linear functional.

Theorem 3.4. *Let A, B be nonempty closed convex subsets of $\mathcal{H}$ such that $A \cap B$ is nonempty and A is bounded. Assume further that for some $f \in S_{\mathcal{H}^*}$ we have*

$$\sup f(B) \leqslant 0 \leqslant \inf f(A).$$

If the pair (A, B) is not regular, then (A, B) is not d-stable.

Proof. We can assume that for any initial point, the alternating projection sequence with respect to the sets A, B converges, because, otherwise, the pair (A, B) is already not d-stable. Therefore, for every $a_0 \in \mathcal{H}$ one has

$$\lim_{n\to\infty} \operatorname{dist}\left(\Pi_n^{A,B}(a_0)\,,\, A\cap B\right) = 0. \tag{3.6}$$

By Lemma 3.3, there exist $\varepsilon_0 > 0$, $r > 0$, a sequence of positive numbers $\{\delta_n\}_n$ with $\delta_n \to 0$ and a sequence $\{w_n\}_{n\geqslant 1} \subset \ker f \cap r\,\mathbb{B}_{\mathcal{H}}$ such that

$$\operatorname{dist}(w_n, A\cap B) > 3\varepsilon_0 \qquad \text{and} \qquad \max\Big\{\operatorname{dist}(w_n, A),\, \operatorname{dist}(w_n, B)\Big\} \leqslant \delta_n \underset{n\to\infty}{\longrightarrow} 0\,.$$

We can assume that $\delta_n \leqslant \varepsilon_0$, for all $n \in \mathbb{N}$. In view of (3.6), the alternating projection sequence starting from w_1 converges to some point $p_1 \in A\cap B$. Let $n_1 \in \mathbb{N}$ be the smallest integer such that

$$\widetilde{p_1} := \Pi_{n_1}^{A,B}(w_1) \in p_1 + \varepsilon_0 \mathbb{B}_{\mathcal{H}}\,.$$

We shall now show that there exist two closed convex sets $\widehat{A}_1$ and $\widehat{B}_1$ which are $3\delta_2$-close (with respect to the Hausdorff distance $\mathrm{D_H}$) to the initial sets A and B, respectively, such that the iterations of the corresponding alternating projection method, starting from $\widetilde{p_1}$, bring us sufficiently close to the point w_2. To this end, notice that $p_1 \in A\cap B \subset \ker f$, $w_2 \in \ker f$ and $w_2 + \delta_2\mathbb{B}_{\mathcal{H}}$ has nontrivial intersection with both sets A and B. Therefore, we can apply Lemma 3.2 for $\delta := \delta_2 \leqslant \varepsilon_0$ to obtain closed convex sets $\widehat{A}_1$ and $\widehat{B}_1$ such that

$$\mathrm{D_H}(A, \widehat{A}_1) \leqslant 3\delta_2\,, \qquad \mathrm{D_H}(B, \widehat{B}_1) \leqslant 3\delta_2 \qquad \text{and} \qquad \lim_{n\to\infty} \Pi_n^{\widehat{A}_1,\widehat{B}_1}(p_1) = w_2.$$

Let now $m_1 \in \mathbb{N}$ be the smallest integer such that

$$\widetilde{w}_2 := \Pi_{m_1}^{A,B}(p_1) \in w_2 + \varepsilon_0\mathbb{B}_{\mathcal{H}}\,.$$

Since $\|p_1 - \widetilde{p_1}\| \leqslant \varepsilon_0$ and the projection operators $P_{\widehat{A}_1}, P_{\widehat{B}_1}$ are non-expansive, we deduce

$$q_2 := \Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(\widetilde{p_1}) \in \widetilde{w}_2 + \varepsilon_0\mathbb{B}_{\mathcal{H}}\,.$$

Consequently, $\|q_2 - w_2\| \leqslant 2\varepsilon_0$ and since $d(w_2, A\cap B) > 3\varepsilon_0$ we get $d(q_2, A\cap B) > \varepsilon_0$. Concatenating the sequences:

$$\{\Pi_n^{A,B}(w_1)\}_{n\leqslant n_1} \quad (\text{joining } w_1 \text{ to } \widetilde{p_1}) \qquad \text{and} \qquad \{\Pi_n^{\widehat{A}_1,\widehat{B}_1}(\widetilde{p_1})\}_{n\leqslant m_1} \quad (\text{joining } \widetilde{p_1} \text{ to } q_2)$$

we obtain a finite sequence of variational alternating projections remaining entirely outside the set $(A\cap B) + \varepsilon_0\mathbb{B}_{\mathcal{H}}$.

We can now iterate this process by considering the alternating projection sequence for the initial sets A and B, starting from the point q_1. In view of (3.6) this sequence converges to some point of the intersection $A \cap B$, that is:

$$\lim_{n\to\infty} \Pi_n^{A,B}(q_1) := p_2 \in A \cap B.$$

Continuing in this way, we generate two sequences $\{A_n\}$ and $\{B_n\}$ of closed convex sets of the form

$$\{A_n\}_{n=1}^{\infty} := \left\{ \underbrace{A,\ldots,A}_{n_1 \text{ times}}, \underbrace{\widehat{A}_1,\ldots,\widehat{A}_1}_{m_1 \text{ times}}, \underbrace{A,\ldots,A}_{n_2 \text{ times}}, \underbrace{\widehat{A}_2,\ldots,\widehat{A}_2}_{m_2 \text{ times}}, \underbrace{A,\ldots,A}_{n_3 \text{ times}}, \underbrace{\widehat{A}_3,\ldots,\widehat{A}_3}_{m_3 \text{ times}}, \ldots \right\}$$

$$\{B_n\}_{n=1}^{\infty} := \left\{ \underbrace{B,\ldots,B}_{n_1 \text{ times}}, \underbrace{\widehat{B}_1,\ldots,\widehat{B}_1}_{m_1 \text{ times}}, \underbrace{B,\ldots,B}_{n_2 \text{ times}}, \underbrace{\widehat{B}_2,\ldots,\widehat{B}_2}_{m_2 \text{ times}}, \underbrace{B,\ldots,B}_{n_3 \text{ times}}, \underbrace{\widehat{B}_3,\ldots,\widehat{B}_3}_{m_3 \text{ times}}, \ldots \right\}$$

respectively, for which the variational alternating projection sequence starting from w_0 and projecting successively onto these sets is not converging. This shows that the pair (A, B) is not d-stable. ■

Remark: Theorem 3.4 still holds in the case where $A \cap B = \emptyset$ and the best approximation sets E and F are nonempty and bounded: in fact, the separation is now provided by the hyperplane which is orthogonal to the displacement vector v (see (2.2)) and a similar procedure to the above can be applied. Moreover, proceeding as in the proof of forthcoming Theorem 3.7, it is possible to omit the assumption that A is bounded.

We shall now establish the main result of our paper in its full generality, removing the additional assumption that the closed convex sets A and B can be separated. For the proof, we need to recall the notion of algebraic interior of a set.

Definition 3.5. Let $C \subseteq \mathcal{H}$ be a nonempty set. The *algebraic interior* of C, denoted by $\operatorname{alg\,int} C$, is the set of all points $c \in C$ such that for every $u \in \mathbb{S}_{\mathcal{H}}$ there exists $\varepsilon_u > 0$ such that $[c, c + \varepsilon_u u) \subseteq C$.

We recall that a set $C \subseteq \mathcal{H}$ is called *absorbing* if for every $x \in \mathcal{H}$, there exists a positive number α such that $x \in tC$ whenever $t > \alpha$. It is easy to see that $x \in \operatorname{alg\,int} C$ if and only if $C - x$ is absorbing. Moreover, if C is a convex set such that $\operatorname{int} C \neq \emptyset$ then, $\operatorname{alg\,int} C = \operatorname{int} C$.

We also need the following well-known corollary of Baire Theorem.

Fact 3.6 (see, e.g., [1, Corollary 3.28])**.** *If a complete metric space is a countable union of closed sets, then at least one of them has a nonempty interior.*

Theorem 3.7. *Let A, B be nonempty closed convex subsets of $\mathcal{H}$ such that $A \cap B$ is nonempty and bounded. Suppose that the pair (A, B) is not regular. Then the pair (A, B) is not d-stable.*

Proof. As in the proof of Theorem 3.4, without loss of generality we may assume that the alternating projection sequence relative to the sets A and B converges for any starting point, that is, for every $a_0 \in \mathcal{H}$, (3.6) holds. We can also assume that $0 \in A \cap B$.

Since the pair (A, B) is not regular and $A \cap B$ is bounded, by [6, Proposition 3.2] the pair (A, B) is not boundedly regular. Therefore, there exist $r > 1$, $\varepsilon \in (0, 1)$ and sequences $\{\delta_n\}_{n\geq 1} \subset (0, \varepsilon/3)$ with $\delta_n \to 0$ and $\{t_n\}_n \subset (r-1)\mathbb{B}_{\mathcal{H}}$ such that:

$$A \cap B \subset (r-1)\mathbb{B}_{\mathcal{H}} \qquad \operatorname{dist}(t_n, A \cap B) \geqslant 3\varepsilon \qquad \delta_n := \max\{\operatorname{dist}(t_n, A), \operatorname{dist}(t_n, B)\}.$$

Let $\{r_h\}_h \subset (r, \infty)$ be a sequence of real numbers such that $r_h \to \infty$ and take an arbitrary point $q_0 \in \mathcal{H}$. Let $n_1 \in \mathbb{N}$ be such that

$$\operatorname{dist}\left(\Pi_{n_1}^{A,B}(q_0),\, A \cap B\right) < \varepsilon.$$

Denote $s_1 = \Pi_{n_1}^{A,B}(q_0)$, let $s_1' \in A \cap B$ be such that $\|s_1 - s_1'\| < \varepsilon$ and set

$$A_1 = A \cap r_1\mathbb{B}_{\mathcal{H}} \qquad \text{and} \qquad B_1 = B.$$

Claim. $0 \notin \operatorname{alg\,int}(A_1 - B_1)$.

Proof of the claim. Let us first observe that since the pair (A_1, B_1) is not regular, by [2, Corollary 4.5], we have $0 \notin \operatorname{int}(A_1 - B_1)$. Notice further that since A_1 is w-compact, the set $A_1 - B_1$ is closed and convex. Let us suppose, towards a contradiction, that $0 \in \operatorname{alg\,int}(A_1 - B_1)$. Then, $A_1 - B_1$ would be an absorbing set and consequently $\cup_{n=1}^{\infty} n(A_1 - B_1) = \mathcal{H}$. In this setting, Fact 3.6 would yield that $\operatorname{int}(A_1 - B_1) \neq \emptyset$. Since $(A_1 - B_1)$ is convex, we would have $0 \in \operatorname{alg\,int}(A_1 - B_1) = \operatorname{int}(A_1 - B_1)$, a contradiction. Therefore, the assertion of the claim holds.

It follows that there exists $u_1 \in S_{\mathcal{H}}$ such that, for every $\theta > 0$, we have $A_1 \cap (B_1 + \theta u_1) = \emptyset$. In particular, the closed convex sets A_1 and $(B_1 + \delta_1 u_1)$ are disjoint and A_1 is w-compact. By the Hahn-Banach theorem there exists $f_1 \in S_{\mathcal{H}^*}$ and $\alpha_1 \in \mathbb{R}$ such that

$$\sup f_1(B_1 + \delta_1 u_1) \leqslant \alpha_1 \leqslant \inf f_1(A_1).$$

Since $s_1' \in A_1$ and $s_1' + \delta_1 u_1 \in (B_1 + \delta_1 u_1)$, there exists $p_1 \in [s_1', s_1' + \delta_1 u_1]$ such that $f_1(p_1) = \alpha_1$. It follows that

$$(p_1 + \delta_1\mathbb{B}_{\mathcal{H}}) \cap A_1 \neq \emptyset, \qquad (p_1 + \delta_1\mathbb{B}_{\mathcal{H}}) \cap (B_1 + \delta_1 u) \neq \emptyset.$$

Similarly, let $a_1 \in A_1$ and $b_1 \in B_1$ be such that $\|a_1 - t_1\| \leqslant \delta_1$ and $\|b_1 - t_1\| \leqslant \delta_1$. Since $b_1 + \delta_1 u_1 \in (B_1 + \delta_1 u_1)$, there exists $w_1 \in [a_1, b_1 + \delta_1 u_1]$ such that $f_1(w_1) = \alpha_1$. Since $\|b_1 + \delta_1 u_1 - t_1\| \leqslant 2\delta_1$, we have $\|w_1 - t_1\| \leqslant 2\delta_1$. Recalling that $\operatorname{dist}(t_1, A_1) < \delta_1$ we obtain

$$(w_1 + 3\delta_1\mathbb{B}_{\mathcal{H}}) \cap A_1 \neq \emptyset \qquad \text{and} \qquad (w_1 + 3\delta_1\mathbb{B}_{\mathcal{H}}) \cap (B_1 + \delta_1 u_1) \neq \emptyset.$$

Applying Lemma 3.2 we obtain two (perturbed) closed convex sets $\widehat{A}_1, \widehat{B}_1$ such that

$$\mathrm{D_H}\left(A_1, \widehat{A}_1\right) \leqslant 9\delta_1 \quad \text{and} \quad \mathrm{D_H}\left(B_1 + \delta_1 u_1, \widehat{B}_1\right) \leqslant 9\delta_1,$$

and such that $\Pi_n^{\widehat{A}_1,\widehat{B}_1}(p_1) \to w_1$ as $n \to \infty$. Notice that

$$\mathrm{D_H}\left(A \cap r_1\mathbb{B}_{\mathcal{H}}, \widehat{A}_1\right) \leqslant 9\delta_1 \quad \text{and} \quad \mathrm{D_H}\left(B, \widehat{B}_1\right) \leqslant 10\delta_1.$$

Since

$$\operatorname{dist}(w_1, A\cap B) \geqslant \operatorname{dist}(t_1, A\cap B) - \|w_1 - t_1\| \geqslant 3\varepsilon - 2\delta_1 > 2\varepsilon + \delta_1,$$

there exists $m_1 \in \mathbb{N}$ such that

$$\operatorname{dist}\left(\Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(p_1), A\cap B\right) > 2\varepsilon + \delta_1.$$

Since $\|p_1 - s_1\| \leqslant \varepsilon + \delta_1$, we have $\|\Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(s_1) - \Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(p_1)\| \leqslant \varepsilon + \delta_1$ and hence

$$\operatorname{dist}\left(\Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(s_1), A\cap B\right) \geqslant \operatorname{dist}\left(\Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(p_1), A\cap B\right) - (\varepsilon+\delta_1) \geqslant \varepsilon.$$

Let us finally set

$$q_1 = \Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(s_1) = \Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}\left(\Pi_{n_1}^{A,B}(q_0)\right).$$

Iterating the same argument as above, we can inductively obtain a sequence of sets $\widehat{A}_h, \widehat{B}_h \subset \mathcal{H}$, together with points $q_h \in \mathcal{H}$ and positive integers n_h, m_h such that, for every $h \in \mathbb{N}$, we have:

(i) $\mathrm{D_H}\left(A \cap r_h\mathbb{B}_{\mathcal{H}}, \widehat{A}_h\right) \leqslant 9\delta_h$ and $\mathrm{D_H}\left(B, \widehat{B}_h\right) \leqslant 10\delta_h$;

(ii) $q_{h+1} = \Pi_{m_h}^{\widehat{A}_h,\widehat{B}_h}\left(\Pi_{n_h}^{A,B}(q_h)\right)$.

To prove that the pair (A,B) is not d-stable, let us consider the sequence $\{A_k\}_k$ of closed convex sets in $\mathcal{H}$ defined by

$$\left\{\underbrace{A,\ldots,A}_{n_1 \text{ times}}, \underbrace{\widehat{A}_1,\ldots,\widehat{A}_1}_{m_1 \text{ times}}, \underbrace{A,\ldots,A}_{n_2 \text{ times}}, \underbrace{\widehat{A}_2,\ldots,\widehat{A}_2}_{m_2 \text{ times}}, \underbrace{A,\ldots,A}_{n_3 \text{ times}}, \underbrace{\widehat{A}_3,\ldots,\widehat{A}_3}_{m_3 \text{ times}}, \ldots\right\}$$

and the the sequence $\{B_k\}_k$ of closed convex sets in $\mathcal{H}$ defined similarly by

$$\left\{\underbrace{B,\ldots,B}_{n_1 \text{ times}}, \underbrace{\widehat{B}_1,\ldots,\widehat{B}_1}_{m_1 \text{ times}}, \underbrace{B,\ldots,B}_{n_2 \text{ times}}, \underbrace{\widehat{B}_2,\ldots,\widehat{B}_2}_{m_2 \text{ times}}, \underbrace{B,\ldots,B}_{n_3 \text{ times}}, \underbrace{\widehat{B}_3,\ldots,\widehat{B}_3}_{m_3 \text{ times}}, \ldots\right\}$$

It follows from (i) and Fact 2.3 that the sequences $\{\widehat{A}_h\}_h$ and $\{\widehat{B}_h\}_h$ Attouch-Wets converge to the sets A and B respectively. Therefore, so do the sequences $\{A_k\}_k$ and $\{B_k\}_k$. Consequently, considering the perturbed alternating projection sequences $\{a_h\}$ and $\{b_h\}$, with respect to $\{A_h\}_h$ and $\{B_h\}_h$ and with starting point q_0, we infer from (ii) that:

$$a_{k_N} = q_{N+1} \qquad \text{where} \quad k_N = \sum_{h=1}^{N} n_h + m_h.$$

Since $k_N \to \infty$ as $N \to \infty$ and $\operatorname{dist}(q_{N+1}, A\cap B) \geqslant \varepsilon$, we deduce that the pair (A,B) is not d-stable. The proof is complete. ■

The above result provides a full proof of Theorem A and answers positively the open problem considered in Problem 1.2.

4. Open Problems and Remarks

We established the equivalence between regularity and d-stability for a pair of closed convex subsets (A, B) of $\mathcal{H}$ provided that $A \cap B$ is nonempty and bounded (or more generally, the best approximation sets E and F are nonempty and bounded). Example 5.2 in [6] shows that regularity does not imply d-stability of the pair (A, B) when $A \cap B$ is unbounded, even in a finite-dimensional setting. The problem of whether d-stability implies or not the regularity of (A, B) remains open.

Another interesting question is to determine if there exists a weaker regularity condition on (A, B) which is equivalent to the norm convergence of the alternating projection method.

The convex feasibility problem for more than two closed convex sets is a rich field of research, where the order (and the frequency) on which the alternating projections are taken would potentially lead to different conclusions and unexplored notions of regularity.

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