Differentiable functions with surjective Clarke Jacobians

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Abstract. We construct, for any $n, m \in \mathbb{N} \setminus \{0\}$, a differentiable locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is $\mathcal{C}^1$ on the complement of an $\mathcal{H}_1$-null set $E \subset \mathbb{R}^n$ and has the property that the range of its limiting Jacobian on E contains the family of all nonempty compact connected sets of $(m\times n)$-matrices. As a consequence, the Clarke Jacobian $\mathrm{J}_c f$ is surjective, that is, its range contains every nonempty compact convex subset of $(m\times n)$-matrices. This reveals a significant difference between differentiable functions and $\mathcal{C}^1$-functions, since for a $\mathcal{C}^1$-function the Clarke Jacobian is always a singleton. As a by-product, we also obtain examples of $\mathcal{C}^1$-smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ (for any $n, m \in \mathbb{N}\setminus\{0\}$) with surjective derivative, that is, $\mathrm{Im}(Df) = \mathbb{R}^{m\times n}$.

Key words. Limiting Jacobian, differentiable Lipschitz function, Whitney extension, Cantor set.

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Contents

1. Main results and state-of-the-art

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called locally Lipschitz if for every $\bar{x} \in \mathbb{R}^n$, there exist $\delta > 0$ and $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\,\|x - y\|, \quad \text{for all } x, y \in B(\bar{x}, \delta), \tag{1}$$

where $\|\cdot\|$ denotes both norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ and $B(\bar{x}, \delta)$ stands for the open ball centered at $\bar{x}$ with radius $\delta > 0$. If (1) holds for all $x, y \in \mathbb{R}^n$ then we say that f is Lipschitz and define its Lipshitz constant $\mathrm{Lip}(f)$ as the infimum of $L > 0$ for which the above inequality holds true.

For any *differentiable* locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$, the *limiting Jacobian* $\mathrm{J}_L f(x)$ of f at $x \in \mathbb{R}^n$ can be defined as follows:

$$\mathrm{J}_L f(x) = \left\{ Q \in \mathbb{R}^{m\times n} : \ \exists \{x_k\}_k \subset \mathbb{R}^n \text{ converging to } x, \ Q = \lim_{k\to+\infty} Df(x_k) \right\}. \tag{2}$$

This gives rise to a multivalued operator $\mathrm{J}_L f : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m\times n}$ which associates to every $x \in \mathbb{R}^n$ a compact subset $\mathrm{J}_L f(x)$ of $\mathbb{R}^{m\times n}$ with $Df(x) \in \mathrm{J}_L f(x)$. Moreover, $\mathrm{J}_L f(x) = \{Df(x)\}$ if and only if the derivative mapping $Df : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is continuous at x. Notice that

$$\mathrm{gph}\,(\mathrm{J}_L f) = \overline{\mathrm{gph}\,(Df)},$$

where $\mathrm{gph}\,(Df)$ denotes the graph of the derivative and $\mathrm{gph}\,(\mathrm{J}_L f) = \{(x,Q) : Q \in \mathrm{J}_L f(x)\}$ stands for the graph of the multivalued map $\mathrm{J}_L f$. The size of $\mathrm{J}_L f(x)$ reflects, in some sense, the degree of discontinuity of the derivative of f.

In this manuscript we construct a differentiable, locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ that is able to represent every nonempty, compact, connected subset of $\mathbb{R}^{m\times n}$ as limiting Jacobian at some point. More precisely, denoting by $\mathcal{H}_1$ the one-dimensional Hausdorff measure and by $\mathcal{K}(\mathbb{R}^{m\times n})$ the set of *nonempty compact connected* subsets of $\mathbb{R}^{m\times n}$ we establish the following result.

Theorem 1.1 (main result)**.** *For any $n, m \in \mathbb{N}\setminus\{0\}$, there exist a differentiable locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ and a subset E of $\mathbb{R}^n$ with $\mathcal{H}_1(E) = 0$, such that*

$$\mathrm{J}_L f(E) := \{\mathrm{J}_L f(x) :\ x \in E\} = \mathcal{K}(\mathbb{R}^{m\times n}) \qquad \textit{and} \qquad f \in C^1(\mathbb{R}^n \setminus E). \tag{3}$$

The above result has a straightforward consequence for the Clarke Jacobian $\mathrm{J}_c f$. We recall that this latter is defined as the convex envelope of the limiting Jacobian, that is,

$$\mathrm{J}_c f(x) := \mathrm{conv}\,(\mathrm{J}_L f(x)), \quad \text{for every } x \in \mathbb{R}^n.$$

Denoting by $\mathcal{K}_{\mathrm{conv}}(\mathbb{R}^{m\times n})$ the set of nonempty compact convex subsets of $\mathbb{R}^{m\times n}$ we obviously have $\mathrm{J}_c f(x) \in \mathcal{K}_{\mathrm{conv}}(\mathbb{R}^{m\times n})$ and we deduce easily from Theorem 1.1 that $\mathrm{J}_c f$ actually takes all of its possible values.

Corollary 1.2 (surjective Clarke Jacobian)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given by Theorem 1.1. Then the Clarke Jacobian map $\mathrm{J}_c f(x) : \mathbb{R}^n \to \mathcal{K}_{\mathrm{conv}}(\mathbb{R}^{m\times n})$ is surjective. In particular,*

$$\mathrm{J}_c f(E) := \{\mathrm{J}_c f(x) :\ x \in E\} = \mathcal{K}_{\mathrm{conv}}(\mathbb{R}^{m\times n}). \tag{4}$$

Surjectivity can also be asserted for the limiting Jacobian map $\mathrm{J}_L f$ in case $m = 1$. Indeed, the derivative Df of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies a Darboux-type property (*c.f.* [9]):

– for every convex compact subset B of $\mathbb{R}^n$ with nonempty interior,

$$\text{the set} \quad Df(B) := \{Df(x') :\ x' \in B\} \quad \text{is connected.}$$

Therefore, if the function f is differentiable and locally Lipschitz, setting $B_k := \overline{B}(x, 1/k)$ (the closed ball of radius $1/k$ and center x) for every $k \geq 1$, the set $\overline{Df(B_k)}$ is nonempty compact connected in $\mathbb{R}^n \equiv \mathbb{R}^{1\times n}$, that is, $\overline{Df(B_k)} \in \mathcal{K}(\mathbb{R}^n)$ and the same is true for the limiting Jabobian $\mathrm{J}_L f(x)$ —also called limiting subdifferential and denoted $\partial_L f(x)$— since it can be written as intersection of nested compact connected sets, that is,

$$\mathrm{J}_L f(x) \equiv \partial_L f(x) = \bigcap_{k\geq 1} \overline{Df(B_k)} \in \mathcal{K}(\mathbb{R}^n).$$

Thus, for $m = 1$, Theorem 1.1 asserts that the limiting Jacobian (subdifferential) $\partial_L f$ is surjective. In particular,

$$\partial_L f(E) = \partial_L f(\mathbb{R}^n) = \mathcal{K}(\mathbb{R}^n), \qquad \text{where } \ \mathcal{H}_1(E) = 0 \quad \text{and} \quad f \in \mathcal{C}^1(\mathbb{R}^n \setminus E). \tag{5}$$

Remark 1.3. Notice that in Theorem 1.1 no surjectivity assertion can be made for the limiting Jacobian if $m \geq 2$, since in this case $J_L f$ may also take disconnected values. To see this, consider the (differentiable Lipschitz) function $g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$t \in \mathbb{R} \mapsto g(t) := \begin{cases} \left(t^2 \sin(t^{-1}), t^2 \cos(t^{-1})\right) & \text{if } t \neq 0, \\ (0,0) & \text{if } t = 0. \end{cases}$$

and notice that $J_L g(0) = \{Q \in \mathbb{R}^{2\times 1} : \ Q_1^2 + Q_2^2 = 1 \text{ or } Q = (0,0)\}$, which is not connected.

Recall that for a $\mathcal{C}^1$-function f we have $\mathrm{J}_c f(x) = \mathrm{J}_L f(x) = \{Df(x)\}$, for all $x \in \mathbb{R}^n$. As a by-product of our approach we obtain the following surjectivity result.

Theorem 1.4 (surjectivity of the derivative of a $\mathcal{C}^1$-function)**.** *For any $n, m \in \mathbb{N}\backslash\{0\}$, there exists a $\mathcal{C}^1$-smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$ with surjective derivative map, that is,*

$$\mathrm{Im}(Df) = \mathbb{R}^{m\times n}.$$

State-of-the-art. It has been recently shown that given any nonempty compact convex (respectively, connected) subset K of $\mathbb{R}^{m\times n}$, there exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ (depending on K) such that $\mathrm{J}_c f(0) = K$ (respectively, $\mathrm{J}_L f(0) = K$), see [2, Theorem 1.1] (respectively, [1, Theorem 1]). Moreover, the function can be taken $\mathcal{C}^\infty$ on $\mathbb{R}^n \diagdown \{0\}$ and differentiable at 0. In addition, exploring symmetries in the construction, the authors of [1, 2] were able to ensure that $\mathrm{J}_c(f|_\mathcal{P})(0) = K|_\mathcal{P}$ (respectively, $\mathrm{J}_L(f|_\mathcal{P})(0) = K|_\mathcal{P}$) for every subspace $\mathcal{P}$ of $\mathbb{R}^n$.

In this setting, Corollary 1.2 (respectively, Theorem 1.1) partially improve the aforementioned results, in the sense that there exists a common differentiable function f which represents all subsets K belonging to the family of interest: nonempty convex (respectively connected) and compact. These sets are now recovered as Clarke (respectively, limiting) Jacobians at some point of an $\mathcal{H}_1$-null subset E of $\mathbb{R}^n$. On the other hand, we can only ensure that f is $\mathcal{C}^1$ on the complement of E (instead of $\mathcal{C}^\infty$) and these representations are no longer stable when intersecting with a subspace.

For the real-valued case (that is, $m = 1$) a locally Lipschitz, differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ has been constructed with the property that every compact connected (respectively, convex) subset of $\mathbb{R}^n$ with *nonempty interior* belongs to the range of the limiting (respectively, Clarke) subdifferential ([5, Theorem 3.16]). The restriction of nonempty interior was an intrinsic limitation of the construction given in [5, Theorem 3.16] (only sets that had nonempty intersection with a predetermined countable dense set could be recovered). In view of Theorem 1.1, this assumption can now be omitted and every nonempty compact connected set can be recovered. In addition, the result holds for every $m \geq 1$, see (3)–(4).

Our approach borrows from [5] the idea of coding the set of interest, which is now the set $\mathcal{K}(\overline{B}_{\mathbb{R}^{m\times n}}) := \{C \in \mathcal{K}(\mathbb{R}^{m\times n}) : C \subset \overline{B}_{\mathbb{R}^{m\times n}}\}$, as a continuous surjective image of the Cantor set Δ. This being said, our current proof differs significantly from the one in [5]. First, the aforementioned coding is now carried out in a particular way that guarantees the existence of a continuous selection. Moreover, we use a specific way to approximate and eventually represent compact connected subsets that contain 0. Finally, the construction uses a version of the Whitney extension theorem, which now replaces the explicit construction given in [5],

Remark 1.5 (alert on the definition)**.** The limiting Jacobian was mainly coined and defined for nonsmooth locally Lipschitz maps, which is particularly relevant in Optimization and in Control theory, see [4, Chapter 2]. In this case the elements of the set $\mathrm{J}_L f(x)$ are used as substitutes for the derivative, whenever this latter does not exist (this only happens in a zero-measured set, due to Rademacher theorem). The original definition of $\mathrm{J}_L f(x)$ in the nonsmooth case differs slightly from the one given in (2). This latter is in fact a simplified equivalent definition valid only for differentiable locally Lipschitz functions.

2. Notation and preliminary results

Notation: $\mathbb{R}^n$ and $\mathbb{R}^m$ are hereby equipped with Euclidean norms denoted indistinctively by $\|\cdot\|$. The space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$ is denoted by $\mathbb{R}^{m\times n}$ and it is equipped with the operator norm: for $Q \in \mathbb{R}^{m\times n}$,

$$\|Q\| := \max\{\|Qx\| : \ \|x\| \leq 1\}.$$

Note that, to simplify the notation, we use the same symbol $\|\cdot\|$ for the norms of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$. We sometimes use the notation $\langle Q, x\rangle$ to refer to the matrix product Qx between $Q \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. Given $a \in \mathbb{R}$, we denote by $(a, 0_{n-1})$ the vector $(a, 0, \ldots, 0) \in \mathbb{R}^n$. For instance, the first canonical vector $e_1 \in \mathbb{R}^n$ is also denoted by $(1, 0_{n-1})$. Vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ will be always denoted with lowercase letters and matrices in $\mathbb{R}^{m\times n}$ with capital letters. Finally, we denote by $B_{\mathbb{R}^n}(x, r)$ (respectively, $B_{\mathbb{R}^{m\times n}}(x, r)$) the open ball of $\mathbb{R}^n$ (respectively, of $\mathbb{R}^{m\times n}$) centered at x with radius $r > 0$. If $x = 0$, we then simplify notation and write $r\mathrm{B}_{\mathbb{R}^n} \equiv B_{\mathbb{R}^n}(0, r)$ and $r\mathrm{B}_{\mathbb{R}^{m\times n}} \equiv B_{\mathbb{R}^{m\times n}}(0, r)$. In particular, $\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}$ stands for the closed unit ball of $\mathbb{R}^{m\times n}$.

Given a nonempty subset B in $\mathbb{R}^{m\times n}$, we denote by $\mathcal{K}(B)$ the family of all nonempty compact connected subsets of B, that is,

$$\mathcal{K}(B) := \{K \subset B : \ K \text{ is nonempty, compact and connected}\} \tag{6}$$

and endow the above set with the Hausdorff distance, that is,

$$D_{\mathrm{H}}(K_1, K_2) := \max\left\{\sup_{x\in K_1} d(x, K_2), \ \sup_{x\in K_2} d(x, K_1)\right\}, \quad \text{for } K_1, K_2 \in \mathcal{K}(B), \tag{7}$$

where $d(x, K) := \inf\{\|x - a\| : a \in K\}$ for every $K \subset \mathbb{R}^n$. It is well-known that if B is compact, the metric space $(\mathcal{K}(B), D_{\mathrm{H}})$ is compact, therefore, there exists a continuous surjective map from the Cantor set Δ to $\mathcal{K}(B)$ ([8, Theorem 4.18] *e.g.*). We shall also consider two particular closed subspaces of $\mathcal{K}(B)$, namely,

$$\mathcal{K}^0(B) := \{K \in \mathcal{K}(B) : \ 0 \in K\} \quad \text{and} \quad \mathcal{K}_{\mathrm{conv}}(B) = \{K \in \mathcal{K}(B) : \ K \text{ convex}\}. \tag{8}$$

Therefore, both $\mathcal{K}^0(B)$ and $\mathcal{K}_{\mathrm{conv}}(B)$ are compact metric spaces. In addition, $(\mathcal{K}_{\mathrm{conv}}(B), D_{\mathrm{H}})$ is also a geodesic space when B is convex. (We refer to [5] for more details.)

In what follows, our objective is to establish the following.

Theorem 2.1. *Let $n, m \in \mathbb{N}\backslash\{0\}$. There are a differentiable Lipschitz function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ with bounded support and a subset E of $\mathbb{R}^n$ with $\mathcal{H}_1(E) = 0$, such that $\Phi \in C^1(\mathbb{R}^n \setminus E)$ and*

$$\mathcal{K}(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}) = \mathrm{J}_L\Phi(E) \subset \mathrm{J}_L\Phi(\mathbb{R}^n) \subset \mathcal{K}(\mathbb{R}^{m\times n}). \tag{9}$$

That is, for any $K \in \mathcal{K}(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$, there exists $x_K \in E$ satisfying $\mathrm{J}_L\Phi(x_K) = K$.

Before we proceed, we observe that Theorem 1.1 is a simple consequence of the above result.

Proof of Theorem 1.1 (assuming Theorem 2.1).
Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be the function given by Theorem 2.1 and denote by $\operatorname{supp}\Phi$ its support. By assumption, $\operatorname{supp}\Phi \subset r\mathrm{B}_{\mathbb{R}^n}$ for some $r > 0$. Let $k \geq 0$. Then the function

$$\Phi_{k,\bar{x}}(x) := k\,\Phi(x - \bar{x}), \quad x \in \mathbb{R}^n,$$

satisfies $\mathcal{K}(k\,\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}) \subset \mathrm{J}_L\Phi_{k,\bar{x}}(\mathbb{R}^n) \subset \mathcal{K}(\mathbb{R}^{m\times n})$ and $\operatorname{supp}\Phi_{k,\bar{x}} \subset B_{\mathbb{R}^n}(\bar{x}, r)$.
Denoting by $e_1 = (1, 0_{n-1}) \in \mathbb{R}^n$ the first canonical vector of $\mathbb{R}^n$, we set $\bar{x}_k := 3kre_1$ and $\Phi_k \equiv \Phi_{k,\bar{x}_k}$ for $k \in \mathbb{N}$. Then

$$\operatorname{supp}\Phi_{k_1} \cap \operatorname{supp}\Phi_{k_2} = \emptyset, \qquad \text{for } k_1 \neq k_2$$

and we can easily check that the function

$$f(x) = \sum_{k=1}^{\infty} \Phi_k(x) = \sum_{k=1}^{\infty} k\,\Phi(x - 3kre_1)$$

satisfies the conclusion of Theorem 1.1. □

Let us first recall a version of Whitney extension theorem for vector-valued functions that we are going to use. It is a special case of [6, Theorem 3.1.14].

Proposition 2.2 (Whitney extension for vector-valued functions)**.** *Let $\Sigma \subset \mathbb{R}^n$ be a nonempty compact set and let $\alpha : \Sigma \to \mathbb{R}^m$ and $\beta : \Sigma \to \mathbb{R}^{m\times n}$ be functions that define a family of 1-jets $\{P_\sigma\}_{\sigma\in\Sigma}$ (polynomials from $\mathbb{R}^n$ to $\mathbb{R}^m$ of degree one) as follows:*

$$P_\sigma(x) = \alpha(\sigma) + \langle \beta(\sigma), x - \sigma\rangle, \quad \textit{for all } x \in \mathbb{R}^n. \tag{10}$$

For $\delta > 0$ consider the quantities:

$$\rho_0(\delta) := \sup_{\sigma_1,\sigma_2\in\Sigma} \left\{ \frac{||P_{\sigma_2}(\sigma_1) - P_{\sigma_1}(\sigma_1)||}{||\sigma_1 - \sigma_2||} :\ 0 < ||\sigma_1 - \sigma_2|| \leq \delta \right\} \tag{11}$$

and

$$\rho_1(\delta) := \sup_{\sigma_1,\sigma_2\in\Sigma} \{||DP_{\sigma_2}(\sigma_1) - DP_{\sigma_1}(\sigma_1)|| :\ 0 < ||\sigma_1 - \sigma_2|| \leq \delta\} \tag{12}$$

and assume that

$$\lim_{\delta\to 0} \rho_0(\delta) \ = \ \lim_{\delta\to 0} \rho_1(\delta) = 0.$$

Then there exists a $\mathcal{C}^1$-smooth function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g|_\Sigma = \alpha$ and $Dg|_\Sigma = \beta$ (that is, the 1-jet P_σ given in (10) *is the first-order Taylor polynomial of g at σ).*

Proof. It follows from [6, Theorem 3.1.14] for the special case where $A = \Sigma$ is compact, $Y = \mathbb{R}^m$ and $k = 1$. □

3. Proof of the main result

In this part, we shall exclusively focus on Theorem 2.1 from which, as we show in the previous section, our main result (Theorem 1.1) follows. Before we give the complete proof of Theorem 2.1, let us outline its main steps for the reader's convenience.

Step 1. There exists $\rho > 1$ such that for every $\varepsilon > 0$ and any finite (not necessarily injective) sequence $A = \{Q_i\}_{i=0}^k \subset \mathbb{R}^{m\times n}$ with $Q_0 \equiv 0$ and $||Q_{i+1} - Q_i|| < \varepsilon$ there exists a $\mathcal{C}^\infty$ function $\Psi_A : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\operatorname{supp}\Psi_A \subset \mathrm{B}_{\mathbb{R}^n} \qquad \text{and} \qquad A \subset \operatorname{Im}(D\Psi_A) \subset A + \rho\varepsilon\,\mathrm{B}_{\mathbb{R}^{m\times n}}.$$

Step 2. For every $C \in \mathcal{K}^0(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$ (compact connected subset of the unit ball with $0 \in C$) there exists a differentiable function $\Psi_C : \mathbb{R}^n \to \mathbb{R}^m$ with bounded support, $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus \{0\}$ such that

$$\mathrm{J}_L \Psi_C(0) = C.$$

Step 3. We codify the (compact metric) set $\mathcal{K}(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$ on the Cantor set Δ in a way that admits a continuous selection, that is, there exists a continuous function $V : \Delta \to \overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}$ and a continuous surjection $h : \Delta \to \mathcal{K}(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$ such that

$$V(t) \in h(t), \qquad \text{for every } t \in \Delta.$$

Step 4. Applying Whitney extension theorem (Proposition 2.2), we obtain a $\mathcal{C}^1$-smooth function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $Dg(t, 0_{n-1}) = V(t)$, for every $t \in \Delta$.

Step 5. We set

$$H(t) := h(t) - V(t) \in \mathcal{K}^0(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}) \qquad \text{and} \qquad [0,1] \setminus \Delta = \bigcup_{i=1}^{\infty} (\ell_i, r_i). \tag{13}$$

Using Step 2, we construct a differentiable Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}^m$ with bounded support such that for every $i \in \mathbb{N}$

$$\mathrm{J}_L \Psi((\ell_i, 0_{n-1})) = H(\ell_i) \qquad \text{and} \qquad \mathrm{J}_L \Psi((r_i, 0_{n-1})) = H(r_i).$$

Moreover, the function Ψ is $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus (\triangle \times \{0\}^{n-1})$ and $D\Psi$ vanishes on $\Delta \times \mathbb{R}^{n-1}$. Therefore, $\mathrm{J}_L \Psi(x) = \{D\Psi(x)\}$, for all $x \in \mathbb{R}^n \setminus (\triangle \times \{0\}^{n-1})$. On the other hand, since $\mathrm{J}_L \Psi$ has a closed graph and H is D_{H}-continuous, we deduce from (13) that

$$\mathrm{J}_L \Psi((t, 0_{n-1})) = H(t), \qquad \text{for every } t \in \Delta.$$

Step 6. The function $\Phi = g + \Psi$ satisfies the conclusion of Theorem 2.1.

We shall now proceed to the proof of Theorem 2.1 according to the above steps. In what follows, we fix $n, m \in \mathbb{N} \setminus \{0\}$.

3.1. **Steps 1–2: representation of any element of $\mathcal{K}^0(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$ as limiting Jacobian.** This part is inspired by the aforementioned works [1, 2]. In particular, the following lemma can be deduced from [1, Lemma 3]. We include a short proof for completeness.

Lemma 3.1 (representing a matrix as Jacobian)**.** *There exists $\rho > 1$ such that for every matrix $Q \in \mathbb{R}^{m\times n}$, there exists a $\mathcal{C}^\infty$-smooth function $\varphi_Q : \mathbb{R}^n \to \mathbb{R}^m$, with* $\mathrm{Lip}(\varphi_Q) \le \rho\|Q\|$ *that satisfies* $\operatorname{supp} \varphi_Q \subset \mathrm{B}_{\mathbb{R}^n}$ *and*

$$\varphi_Q(x) = Qx, \qquad \textit{for all } x \in 2^{-1}\,\mathrm{B}_{\mathbb{R}^n}.$$

Proof. Take any $\mathcal{C}^\infty$-smooth bump function $\phi : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp} \phi \subset \mathrm{B}_{\mathbb{R}^n}$ and $\phi|_{2^{-1}\mathrm{B}_{\mathbb{R}^n}} \equiv 1$ and set $\varphi_Q(x) := \phi(x)\,Qx$, for all $x \in \mathbb{R}^n$. Indeed, φ_Q is $\rho\|Q\|$-Lipschitz, with $\rho = \|\phi\|_\infty + \|D\phi\|_\infty$. $\square$

From now on, we fix $\rho > 1$ to be the constant given by Lemma 3.1. From the above lemma, we readily get the following result.

Lemma 3.2 (recovering a discrete ε-path by Jacobians)**.** *Let $\varepsilon > 0$ and $A := \{Q_k\}_{k=0}^{\ell} \subset \mathbb{R}^{m\times n}$ be a finite sequence of matrices (with possible repetitions) such that, $Q_0 = 0$ and $\|Q_{k+1} - Q_k\| \leq \varepsilon$ for all $k \in \{0, ..., \ell-1\}$. Then, there exists a $\mathcal{C}^\infty$-function $\varphi_A : \mathbb{R}^n \to \mathbb{R}^m$, with $\operatorname{supp}\varphi_A \subset \mathrm{B}_{\mathbb{R}^n}$, such that*

$$\{Q_k\}_{k=0}^{\ell} \subset \mathrm{Im}(D\varphi_A) \subset \{Q_k\}_{k=0}^{\ell} + \rho\varepsilon\, \mathrm{B}_{\mathbb{R}^n}.$$

Proof. Let $\varphi_k := \varphi_{Q_{k+1}-Q_k}$ be the $\mathcal{C}^\infty$-function obtained in Lemma 3.1 corresponding to the matrix $Q := Q_{k+1} - Q_k$, for $k \in \{0, ..., \ell-1\}$. Clearly, $\mathrm{Lip}(\varphi_k) \leq \rho\varepsilon$. Notice that the $\mathcal{C}^\infty$-smooth function

$$\tilde{\varphi}_k(x) = 2^{-2k}\, \varphi_k(2^{2k}\, x), \quad x \in \mathbb{R}^n,$$

satisfies $\operatorname{supp}\tilde{\varphi}_k \subset 2^{-2k}\, \mathrm{B}_{\mathbb{R}^n}$ and $\tilde{\varphi}_k(x) = (Q_{k+1} - Q_k)x$, for every $x \in 2^{-(2k+1)}\, \mathrm{B}_{\mathbb{R}^n}$, that is,

$$D\tilde{\varphi}_k\,|_{2^{-(2k+1)}\,\mathrm{B}_{\mathbb{R}^n}} \equiv Q_{k+1} - Q_k, \qquad \text{for every} \quad k \in \{0, ..., \ell-1\}\,.$$

We now define the $\mathcal{C}^\infty$-function $\varphi_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\varphi_A(x) = \sum_{k=0}^{\ell-1} \tilde{\varphi}_k(x) \equiv \sum_{k=0}^{\ell-1} 2^{-2k}\, \varphi_k(2^{2k} x), \text{ for all } x \in \mathbb{R}^n.$$

Since

$$\sum_{k=0}^{j} D\tilde{\varphi}_k(x) = Q_{j+1}, \quad \text{for all } x \in 2^{-(2j+1)}\, \mathrm{B}_{\mathbb{R}^n} \quad \text{and } j \in \{0, ..., \ell-1\}\,,$$

we easily conclude that for every $j \in \{0, ..., \ell-1\}$ and every $x \in \big(2^{-(2j+1)}\, \mathrm{B}_{\mathbb{R}^n}\big) \setminus \big(2^{-(2j+2)} \mathrm{B}_{\mathbb{R}^n}\big)$ we have $D\varphi_A(x) = \sum_{k=0}^{\ell-1} D\tilde{\varphi}_k(x) = Q_{j+1}$. The assertion of the statement follows easily. $\square$

The following lemma provides adequate approximations of compact connected sets by ε-paths (which are representable by the previous lemma). This is an intermediate step to deduce their representability by a limiting Jacobian.

Lemma 3.3 (approximating compact connected sets by ε-paths)**.** *Let $K \subset \mathbb{R}^{m\times n}$ be compact, connected such that $0 \in K$ and let $\varepsilon > 0$. Then, for any finite ε-net $A \subset K$, with $0 \in A$, there exists a surjective (not necessarily injective) enumeration $\{Q_k\}_{k=0}^{\ell}$ of the elements of A such that*

$$Q_0 = 0 \quad \textit{and} \quad \|Q_{k+1} - Q_k\| \leq 3\varepsilon, \textit{ for all } k \in \{0, ..., \ell-1\}\,.$$

Proof. It follows from the fact that K is a compact connected set and that $K \subset \cup_{Q\in A} B(Q, 2\varepsilon)$. $\square$

In the next lemma we construct functions with controlled oscillation and with two prescribed limiting Jacobians.

Lemma 3.4 (recovering two compact connected sets by Jacobians)**.** *Let $0 \leq \ell < r \leq 1$, $0 < \varepsilon < 1$ and $C_\ell, C_r \in \mathcal{K}^0(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$ (compact connected subsets of $\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}$ containing 0). Then there exists a differentiable, Lipschitz function $\psi \equiv \psi_{\ell,r} \colon \mathbb{R}^n \to \mathbb{R}^m$, which is $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus \{(\ell, 0_{n-1}), (r, 0_{n-1})\}$, such that*

- *(i)* $\operatorname{supp}\psi \subset \bigcup_{t\in[\ell,r]} \overline{B}_{\mathbb{R}^n}((t, 0_{n-1}), \rho_t)$ *where* $\rho_t := 2^{-1} \min\{t - \ell, r - t\}$.
- *(ii)* $D\psi(\mathbb{R}^n) \subset (C_\ell \cup C_r) + \varepsilon\, \mathrm{B}_{\mathbb{R}^{m\times n}}$.
- *(iii)* $\mathrm{J}_L\psi(\ell, 0_{n-1}) = C_\ell$ *and* $\mathrm{J}_L\psi(r, 0_{n-1}) = C_r$.

Moreover, for any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we have

$$\|\psi(x)\| \,\leq\, \min\left\{|\ell - x_1|^2,\, |r - x_1|^2\right\}. \tag{14}$$

Proof. Consider two sequences $\{c_j\}_j \subset \left[\ell, \ell + \dfrac{r-\ell}{3}\right]$ and $\{c'_j\}_j \subset \left[r - \dfrac{r-\ell}{3}, r\right]$ defined as follows:

$$c_j = \ell + \frac{1}{j+1}\left(\frac{r-\ell}{3}\right) \quad \text{and} \quad c'_j = r - \frac{1}{j+1}\left(\frac{r-\ell}{3}\right), \text{ for all } j \in \mathbb{N}. \tag{15}$$

Then $\{c_j\}_j$ converges decreasingly to ℓ and $\{c'_j\}_j$ converges increasingly to r. For any $j \in \mathbb{N}$, let $A_j \subset C_\ell$ and $A'_j \subset C_r$ be finite $\frac{\varepsilon}{3\rho(j+1)}$-nets of C_ℓ and C_r respectively. Then, combining Lemma 3.3 and Lemma 3.2 we deduce that there exist $\mathcal{C}^\infty$-functions $\varphi_{A_j} : \mathbb{R}^n \to \mathbb{R}^m$ and $\varphi_{A'_j} : \mathbb{R}^n \to \mathbb{R}^m$ with support contained in $\mathrm{B}_{\mathbb{R}^n}$ such that

$$A_j \subset \mathrm{Im}(D\varphi_{A_j}) \subset A_j + \left(\frac{\varepsilon}{j+1}\right)\mathrm{B}_{\mathbb{R}^{m\times n}} \quad \text{and} \quad A'_j \subset \mathrm{Im}(D\varphi_{A'_j}) \subset A'_j + \left(\frac{\varepsilon}{j+1}\right)\mathrm{B}_{\mathbb{R}^{m\times n}}. \tag{16}$$

Take a decreasing sequence $\{\delta_j\}_{j\geq 1} \subset (0,1)$ converging to 0, with the property that the intervals

$$I_j := (c_j - \delta_j, c_j + \delta_j), \quad j \in \mathbb{N}$$

are mutually disjoint. For this, it is sufficient to have

$$\delta_j + \delta_{j+1} \leq 2\delta_j \leq \frac{r-\ell}{3(j+2)^2} < \left(\frac{r-\ell}{3}\right)\left(\frac{1}{j+1} - \frac{1}{j+2}\right). \tag{17}$$

We shrink further the values of $\{\delta_j\}_{j\geq 1}$ if necessary, to ensure that

$$\delta_j \max\left\{\|\varphi_{A_j}\|_\infty, \|\varphi_{A'_j}\|_\infty\right\} \leq \min\left\{c_j - \delta_j - \ell,\, r - c'_j - \delta_j\right\}^2. \tag{18}$$

Then we set for every $j \geq 1$ and $x \in \mathbb{R}^n$

$$\tilde{\varphi}_j(x) = 2^{-1}\delta_j\ \varphi_{A_j}(2\delta_j^{-1}(x - c_j)) \qquad \text{and} \qquad \tilde{\varphi}'_j(x) = 2^{-1}\delta_j\ \varphi_{A'_j}(2\delta_j^{-1}(x - c'_j)).$$

Notice that, thanks to (17), the functions $\{\tilde{\varphi}_j\}_j \cup \{\tilde{\varphi}'_j\}_j$ are $\mathcal{C}^\infty$ and have pairwise disjoint supports.

It follows that the function $\psi : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\psi(x) := \sum_{j=0}^{\infty}\left(\tilde{\varphi}_j(x) + \tilde{\varphi}'_j(x)\right)$$

is Lipschitz, $\mathcal{C}^\infty$-smooth on $\mathbb{R}^n \setminus \{(\ell, 0_{n-1}), (r, 0_{n-1})\}$ and satisfies (i). Thanks to (18), we have

$$\|\psi(x)\| \leq \min\left\{|\ell - x_1|^2,\, |r - x_1|^2\right\},$$

which ensures that ψ is differentiable at both points $(\ell, 0_{n-1})$ and $(r, 0_{n-1})$ with derivative

$$D\psi(\ell, 0_{n-1}) = D\psi(r, 0_{n-1}) = 0.$$

We deduce easily from (16) that ψ satisfies (ii). Finally, since

$$\lim_{j\to 0} \max\left\{d_H(A_j, C_\ell),\, d_H(A'_j, C_r)\right\} = 0,$$

and (16), we deduce that

$$\mathrm{J}_L\psi(\ell, 0_{n-1}) = C_\ell \qquad \text{and} \qquad \mathrm{J}_L\psi(r, 0_{n-1}) = C_r.$$

The proof is complete. □

3.2. **Step 3: a coding of $\mathcal{K}(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$ that admits a continuous selection.** We first recall that if the set B is nonempty and compact, then the metric space $(\mathcal{K}(B), D_H)$ is compact, where D_H denotes the canonical Hausdorff distance.

Lemma 3.5 (coding with a continuous selection)**.** *Let $B \subset \mathbb{R}^{m\times n}$ be a nonempty compact set. Then there exists a continuous surjective function $h : \triangle \to (\mathcal{K}(B), D_H)$ that admits a continuous selection. That is, there is a continuous function $V : \triangle \to B$ such that*

$$V(t) \in h(t), \quad \textit{for all } t \in \triangle.$$

Proof. Since $(\mathcal{K}(B), D_H)$ is a compact metric space, there exists a continuous surjective function $h_1 : \triangle \to \mathcal{K}(A)$. Since h_1 is continuous with respect to the Hausdorff distance and B is compact, the set

$$\mathcal{A} := \bigcup_{t\in\triangle} (\{t\} \times h_1(t)) \;=\; \big\{(t, Q) \in \mathbb{R} \times \mathbb{R}^{m\times n} :\; t \in \triangle,\; Q \in h_1(t)\big\}$$

is compact and consequently, there exists a continuous surjective function $h_2 : \triangle \to \mathcal{A}$. Let now π_1 and π_2 denote the canonical projections of $\mathbb{R} \times \mathbb{R}^{m\times n}$ onto $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ respectively and define the (continuous) functions

$$h = h_1 \circ \pi_1 \circ h_2 \quad \text{and} \quad V = \pi_2 \circ h_2.$$

It follows easily that h is surjective and that $V(t) \in h(t)$, for all $t \in \triangle$. □

3.3. **Steps 4–6: construction of the function.** Let $\{\ell_k\}_k$ and $\{r_k\}_k$ be two injective sequences such that

$$[0, 1]\backslash\triangle = \bigcup_{k=0}^{\infty} (\ell_k, r_k) \qquad \text{(disjoint union of open intervals)}. \tag{19}$$

Let us consider the functions

$$h : \triangle \to \mathcal{K}(\overline{B}_{\mathbb{R}^{n\times m}}(0, 1)) \qquad \text{and} \qquad V : \triangle \to \overline{B}_{\mathbb{R}^{n\times m}}(0, 1)$$

given by Lemma 3.5. We extend the selection V (defined on Δ) to a function $\hat{V} : [0, 1] \to \mathbb{R}^m$ by linear interpolation:

$$\hat{V}(t) = \left(\frac{r_k - t}{r_k - \ell_k}\right) V(\ell_k) + \left(\frac{t - \ell_k}{r_k - \ell_k}\right) V(r_k), \quad \text{for all } t \in (\ell_k, r_k). \tag{20}$$

Notice that $\hat{V}$ is a continuous extension of V from Δ to $[0, 1]$. We set

$$\Sigma := [0, 1] \times \{0\}_{n-1}$$

and define the function

$$\begin{cases} \alpha : \Sigma \to \mathbb{R}^m \\ \alpha(\sigma) \equiv \alpha\left((t, 0_{n-1})\right) = \int_0^t \langle \hat{V}(s), e_1\rangle\, ds, \qquad \text{for every } \sigma = (t, 0_{n-1}) \in \Sigma, \end{cases} \tag{21}$$

where $e_1 = (1, 0_{n-1})$. We also define

$$\begin{cases} \beta : \Sigma \to \mathbb{R}^{m\times n} \\ \beta(\sigma) \equiv \beta\left((t, 0_{n-1})\right) = \hat{V}(t). \end{cases} \tag{22}$$

We aim to apply Proposition 2.2 (Whitney extension theorem) and show that the function $\alpha:\Sigma\to\mathbb{R}^m$ admits a $\mathcal{C}^1$-smooth extension $g:\mathbb{R}^n\to\mathbb{R}^m$ satisfying $Dg(t,0_{n-1})=\hat{V}(t)$, for all $\sigma=(t,0_{n-1})\in\Sigma$. To this end, we take $0\le t_1,t_2\le 1$ and set $\sigma_i=(t_i,0_{n-1})\equiv t_ie_1$, $i\in\{1,2\}$. Recalling (10) we have:

$$P_{\sigma_i}(x)=\alpha(\sigma_i)+\langle\beta(\sigma_i),x-\sigma_i\rangle=\int_0^{t_i}\langle\hat{V}(s),e_1\rangle\,ds\;+\;\langle\hat{V}(t_i),x-\sigma_i\rangle\,,$$

and consequently, in view of (21),

$$\begin{aligned}P_{\sigma_2}(\sigma_1)-P_{\sigma_1}(\sigma_1)&=\int_{t_1}^{t_2}\langle\hat{V}(s),e_1\rangle\,ds\;+\;\Big\langle\hat{V}(t_2),\,\underbrace{t_1e_1}_{\sigma_1}-\underbrace{t_2e_1}_{\sigma_2}\Big\rangle\\ &=\int_{t_1}^{t_2}\langle\hat{V}(s)-\hat{V}(t_2),e_1\rangle\,ds \qquad (23)\end{aligned}$$

Denoting by T the interval $[t_1,t_2]$, if $t_1\le t_2$ or the interval $[t_2,t_1]$, if $t_1>t_2$, we obtain

$$\|P_{\sigma_2}(\sigma_1)-P_{\sigma_1}(\sigma_1)\|\;\le\;\underbrace{|t_2-t_1|}_{=\|\sigma_1-\sigma_2\|}\sup_{s\in T}\|\hat{V}(s)-\hat{V}(t_2)\|. \qquad (24)$$

Notice that since $\hat{V}$ is uniformly continuous on the compact set Σ it holds:

$$\rho_1(\delta):=\sup_{\|\sigma_1-\sigma_2\|\le\delta}\|\beta(\sigma_1)-\beta(\sigma_2)\|\underbrace{=}_{(22)}\sup_{|t_1-t_2|\le\delta}\left\|\hat{V}(t_1)-\hat{V}(t_2)\right\|\to 0\;\;(\text{as }\delta\to 0).$$

We also deduce from (24) that

$$\rho_0(\delta):=\sup_{\|\sigma_1-\sigma_2\|\le\delta}\left\{\frac{\|P_{\sigma_2}(\sigma_1)-P_{\sigma_1}(\sigma_1)\|}{\|\sigma_1-\sigma_2\|}\right\}\underbrace{\le}_{(24)}\sup_{|t_1-t_2|\le\delta}\left\|\hat{V}(t_1)-\hat{V}(t_2)\right\|\to 0\;\;(\text{as }\delta\to 0).$$

Applying Proposition 2.2 we obtain a $\mathcal{C}^1$-function $g:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$Dg((t,0_{n-1}))=\hat{V}(t),\;\text{ for all }t\in\triangle.$$

Multiplying, if necessary, g by a smooth cut-off function which is equal to 1 on $\overline{\mathrm{B}}_{\mathbb{R}^n}$, we can assume, with no loss of generality, that g is a Lipschitz $\mathcal{C}^1$-function with bounded support.

We shall now modify g in the cylinders $(\ell_k,r_k)\times\mathbb{R}^{n-1}\subset\mathbb{R}^n$. To this end, for any $k\in\mathbb{N}$, set

$$H(\ell_k):=h(\ell_k)-V(\ell_k)\qquad\text{and}\qquad H(r_k):=h(r_k)-V(r_k)$$

and consider the Lipschitz function $\psi_k:\mathbb{R}^n\to\mathbb{R}^m$ given by Lemma 3.4 for $\ell=\ell_k$, $r=r_k$ and $\varepsilon=(k+1)^{-1}$ and the sets $C_{\ell_k}=H(\ell_k)$ and $C_{r_k}=H(r_k)$ in $\mathcal{K}^0(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$. We finally set

$$\begin{cases}\Phi:\mathbb{R}^n\to\mathbb{R}^m\\ \Phi(x):=g(x)+\sum_{k=0}^{\infty}\psi_k(x),\quad\text{for all }x\in\mathbb{R}^n.\end{cases}$$

We shall show that Φ satisfies the assertion of Theorem 2.1. Set $\Psi:=\sum_{k=0}^{\infty}\psi_k$. Let us first recall that $g\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^m)$ and that each function ψ_k is $\mathcal{C}^\infty$ on $\mathbb{R}^n\setminus\{(\ell_k,0_{n-1}),\,(r_k,0_{n-1})\}$ and differentiable everywhere. Since the supports of the functions $\{\psi_k\}_k$ are pairwise disjoint and satisfy Lemma 3.4 (i), it follows that Ψ is Lipschitz and $\mathcal{C}^1$ on $\mathbb{R}^n\setminus(\triangle\times\{0_{n-1}\})$, with $D\Psi(x)=D\psi_k(x)$ for all $x\in(\ell_k,r_k)\times\mathbb{R}^{n-1}$ and $D\Psi(x)=0$ for all $x\in(\triangle\times\mathbb{R}^n)\setminus(\triangle\times\{0\}^n)$. Moreover, combining with (14) we deduce that

$$\|\Psi(x)\|\;=\;\left\|\sum_{k=0}^{\infty}\psi_k(x)\right\|\;\le\;\inf_{t\in\{l_k\}_k\bigcup\{r_k\}_k}\left\{|t-x_1|^2\right\}=\;\min_{t\in\triangle}\left\{|t-x_1|^2\right\}$$

and consequently Ψ is everywhere differentiable with

$$D\Psi(x) = 0, \text{ for all } x \in \triangle \times \{0\}^{n-1}.$$

Thanks to Lemma 3.4 (iii), we deduce that $J_L\Psi(\ell_k, 0_{n-1}) \supset H(\ell_k)$ and $J_L\Psi(r_k, 0_{n-1}) \supset H(r_k)$ for all $k \in \mathbb{N}$. Since the graph of the limiting Jacobian $J_L\Psi$ is closed and H is continuous for the Hausdorff distance, we get that

$$J_L\Psi(t, 0_{n-1}) \supset H(t), \quad \text{for all } t \in \triangle.$$

We shall now prove that the above inclusion is in fact equality, that is, we have $J_L\Psi(t, 0_{n-1}) = H(t)$, for every $t \in \triangle$. Let us first consider the case $t \in \triangle \setminus \{\ell_k, r_k : k \in \mathbb{N}\}$ and fix $\varepsilon > 0$. Since H is continuous, there exists $\delta > 0$ such that

$$D_H(H(t), H(s)) \leq \frac{\varepsilon}{2}, \quad \text{for all } s \in \triangle \cap (t - \delta, t + \delta).$$

Shrinking δ if needed, we can also ensure that if $(\ell_k, r_k) \cap (t - \delta, t + \delta) \neq \emptyset$ for some $k \in \mathbb{N}$, then

$$\max\left(D_H(H(t), H(\ell_k)),\ D_H(H(t), H(r_k))\right) \leq \frac{\varepsilon}{2}.$$

Set $N(\delta) := \inf\{k \in \mathbb{N} :\ (\ell_k, r_k) \cap (t - \delta, t + \delta) \neq \emptyset\}$. Clearly, $N(\delta)$ tends to $+\infty$ as δ tends to 0. We deduce from Lemma 3.4 (ii) that

$$D\Psi((\ell_k, r_k) \times \mathbb{R}^{n-1}) \equiv D\psi_k((\ell_k, r_k) \times \mathbb{R}^{n-1}) \subset (H(\ell_k) \cup H(r_k)) + (k+1)^{-1} B_{\mathbb{R}^{m\times n}}, \text{ for all } k \in \mathbb{N}.$$

Shrinking further δ to ensure that $N(\delta) + 1 > 2\varepsilon^{-1}$ and recalling that $D\Psi \equiv 0$ on $\triangle \times \mathbb{R}^{n-1}$, we obtain that

$$D\Psi((t - \delta, t + \delta) \times \mathbb{R}^{n-1}) \subset H(t) + \varepsilon B_{\mathbb{R}^{m\times n}}.$$

Therefore, $J_L\Psi((t, 0_{n-1})) \subset H(t) + \varepsilon B_{\mathbb{R}^{m\times n}}$. Since $\varepsilon > 0$ is arbitrary, we finally conclude that $J_L\Psi((t, 0_{n-1})) \subset H(t)$. So, $J_L\Psi((t, 0_{n-1})) = H(t)$.

We now consider the case $t \in \{\ell_k\}_{k\in\mathbb{N}} \cup \{r_k\}_{k\in\mathbb{N}}$ and focus on the formula of $\mathrm{J}_L\Psi(t, 0_{n-1})$ given by (2). Let us assume, to fix the ideas, that $t = \ell_{k_0}$ for some $k_0 \in \mathbb{N}$ (the case $t = r_{k_0}$ can be treated analogously). Then every element Q of $\mathrm{J}_L\Psi(t, 0_{n-1})$ can be obtained as limit of a sequence of derivatives of Ψ lying either on the cylinder $[\ell_k, r_k] \times \mathbb{R}^{n-1}$ (where $\Psi \equiv \psi_{k_0}$) or a the cylinder of the form $(\ell - \delta, \ell) \times \mathbb{R}^{n-1}$ (for any $\delta > 0$ arbitrary small). In the first case, we can directly use Lemma 3.4 (iii) while in the second case the analysis of the set of limits follows the same steps as above. The details are left to the reader.

Finally, since the function g is $\mathcal{C}^1$-smooth, we deduce that for every $t \in \triangle$

$$\mathrm{J}_L\Phi(t, 0_{n-1}) = Dg(t, 0_{n-1}) + \mathrm{J}_L\Psi(t, 0_{n-1}) = V(t) + H(t) = h(t).$$

On the other hand, for any $x \in \mathbb{R}^n \setminus (\triangle \times \{0\}^n)$ we have

$$\mathrm{J}_L\Phi(x) = \{Dg(x) + D\Psi(x)\} \in \mathcal{K}(\mathbb{R}^{m\times n}).$$

The proof is now complete. □

3.4. **A simple proof for the case $n = m = 1$.** In this section we illustrate the above proof for the case $n = m = 1$. In this one-dimensional case, the proof is much simpler: any coding of the family of the closed intervals of $[-1, 1]$ admits a natural continuous selection (the mid-point of the interval) and a full construction can be carried out, without passing through a Whitney extension result. We point out that some extra considerations should be taken into account in order to construct a 1-Lipschitz function. This result was also obtained in [5, Theorem 3.7] with a much more elaborated proof.

Theorem 3.6 (case $n = m = 1$). *There exists a differentiable* 1*-Lipschitz function* $f : \mathbb{R} \to \mathbb{R}$ *with bounded support, such that for any nonempty closed interval* $[a, b] \subset [-1, 1]$*, there is a point* x *such that* $\mathrm{J}_L f(x) = [a, b]$.

Proof. Let us set

$$\mathcal{T} := \{(a, b) \in \mathbb{R}^2 : \ -1 \leq a \leq b \leq 1\}.$$

(Notice the natural correspondence between $\mathcal{T}$ and $\mathcal{K}([-1, 1])$: for $(a, b) \in \mathcal{T}$ we associate the closed interval $[a, b]$ of $[-1, 1]$. We further equip $\mathcal{T}$ with the distance inherited by the infinite norm $\|\cdot\|_\infty$ on $\mathbb{R}^2$, so that $\mathcal{T}$ becomes isometrically isomorphic to $\mathcal{K}([-1, 1])$.) Since $\mathcal{T}$ is compact, there exists a continuous surjection $h : \triangle \to \mathcal{T}$. We may also assume that $h(0) = h(1) = (0, 0)$. Let $\{\ell_k\}_k$ and $\{r_k\}_k$ be the injective sequences defined in (19), that is,

$$[0, 1] \backslash \triangle = \bigcup_{k \in \mathbb{N}} (\ell_k, r_k).$$

We first extend h from Δ to $[0, 1]$ by linear interpolation, that is, for every $k \in \mathbb{N}$ we have:

$$h(t) =: \left(\frac{r_k - t}{r_k - \ell_k}\right) h(\ell_k) \ + \ \left(\frac{t - \ell_k}{r_k - \ell_k}\right) h(r_k), \quad \text{for all } t \in (\ell_k, r_k).$$

We further extend h to the whole line $\mathbb{R}$ by setting $h(t) \equiv (0, 0)$ for $t \in \mathbb{R} \smallsetminus [0, 1]$.

We write $h(t) := (h_1(t), h_2(t))$, for every $t \in \mathbb{R}$. Then, the function $V : \mathbb{R} \to [-1, 1]$ defined by $V(t) = 2^{-1}(h_1(t) + h_2(t))$ is a continuous selection of the multivalued map $H : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $H(t) = [h_1(t), h_2(t)]$. We now define the function

$$\begin{cases} a : \mathbb{R} \to \mathbb{R} \\ a(t) := h_2(t) - V(t) = \dfrac{h_2(t) - h_1(t)}{2} \geq 0, \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function such that

$$\operatorname{supp} \phi \subset [-1, 1] \qquad \text{and} \qquad D\phi(\mathbb{R}) = [-1, 1].$$

For each $k \in \mathbb{N}$, we consider sequences $\{c_{k,j}\}_j$, $\{c'_{k,j}\}_j \subset (\ell_k, r_k)$ defined as in (15), that is,

$$c_{k,j} = \ell_k + \frac{1}{j+1}\left(\frac{r_k - \ell_k}{3}\right) \quad \text{and} \quad c'_{k,j} = r_k - \frac{1}{j+1}\left(\frac{r_k - \ell_k}{3}\right), \text{ for all } j \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, let $\{\delta_{k,j}\}_j \subset (0, 1)$ be a decreasing sequence, converging to 0 as $j \to \infty$ and satisfying

$$\delta_{k,j} \|\phi\|_\infty \max\left(a(c_{k,j}), a(c'_{k,j})\right) \leq \left(\min\{c_{k,j} - \delta_{k,j} - \ell_k, \ r_k - c_{k,j} - \delta_{k,j}\}\right)^2, \tag{25}$$

and be such that the family of intervals

$$\left\{[c_{k,j} - \delta_{k,j}, c_{k,j} + \delta_{k,j}],\ [c'_{k,j} - \delta_{k,j}, c'_{k,j} + \delta_{k,j}] : \quad k, j \in \mathbb{N}\right\}$$

is pairwise disjoint. Consider further sequences $\{\varepsilon_{k,j}\}_{k,j}$ and $\{\varepsilon'_{k,j}\}_{k,j}$ defined by

$$\varepsilon_{k,j} = \frac{a(c_{k,j})}{j+1} \qquad \text{and} \qquad \varepsilon'_{k,j} = \frac{a(c'_{k,j})}{j+1}.$$

Note that

$$\lim_{j\to\infty} \varepsilon_{k,j} = \lim_{j\to\infty} \varepsilon'_{k,j} = 0, \quad \text{for all } k \in \mathbb{N}. \tag{26}$$

Since V is continuous, shrinking further $\delta_{k,j}$ if necessary, we may also assume that

$$\begin{aligned} &V(t) + [-a(c_{k,j}) + \varepsilon_{k,j}, a(c_{k,j}) - \varepsilon_{k,j}] \subset [-1,1], \quad \text{for all } t \in (c_{k,j} - \delta_{k,j}, c_{k,j} + \delta_{k,j}), \\ \text{and } &V(t) + [-a(c'_{k,j}) + \varepsilon'_{k,j}, a(c'_{k,j}) - \varepsilon'_{k,j}] \subset [-1,1], \quad \text{for all } t \in (c'_{k,j} - \delta_{k,j}, c'_{k,j} + \delta_{k,j}). \end{aligned} \tag{27}$$

Consider the functions $\psi_{k,j} : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_{k,j}(t) := \delta_{k,j} \Big((a(c_{k,j}) - \varepsilon_{k,j})\, \phi(\delta_{k,j}^{-1}(t - c_{k,j})) + (a(c'_{k,j}) - \varepsilon'_{k,j})\, \phi(\delta_{k,j}^{-1}(t - c'_{k,j})) \Big).$$

Observe that $\operatorname{supp} \psi_{k,j} \subset \{c_{k,j}, c'_{k,j}\} + [-\delta_{k,j}, \delta_{k,j}]$, and

$$\begin{aligned} D\psi_{k,j}\left([c_{k,j} - \delta_{k,j}, c_{k,j} + \delta_{k,j}]\right) &= [-a(c_{k,j}) + \varepsilon_{k,j}, a(c_{k,j}) - \varepsilon_{k,j}], \\ D\psi_{k,j}\left([c'_{k,j} - \delta_{k,j}, c'_{k,j} - \delta_{k,j}]\right) &= [-a(c'_{k,j}) + \varepsilon'_{k,j}, a(c'_{k,j}) - \varepsilon'_{k,j}]. \end{aligned} \tag{28}$$

We finally set

$$\begin{cases} \Psi : \mathbb{R} \to \mathbb{R} \\ \Psi(t) := \sum_{k,j=0}^{\infty} \psi_{k,j}(t). \end{cases}$$

Note that, for every $t \in \mathbb{R}$, there is at most one couple (k,j) such that $t \in \operatorname{supp} \psi_{k,t}$. As we did in the proof of Theorem 2.1, we can show that $D\Psi(t) = \sum_{k,j=0}^{\infty} D\psi_{k,j}(t)$ for all $t \in \mathbb{R} \setminus \triangle$. Thanks to (25), it follows that $|\Psi(t)| = \min\{(t-x)^2 : \ x \in \triangle\}$. Therefore, $D\Psi(t) = 0$ for all $t \in \triangle$. Hence, combining with (26), (28) and the continuity of a, we get

$$J_L\Psi(t) = [-a(t), a(t)] \equiv \left[\frac{h_1(t) - h_2(t)}{2}, \frac{h_2(t) - h_1(t)}{2}\right], \quad \text{for all } t \in \triangle.$$

We now consider the $\mathcal{C}^1$-function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) := \int_0^t V(s)ds = \int_0^t \frac{h_1(s) + h_2(s)}{2} ds, \quad \text{for all } t \in \mathbb{R}.$$

Since V is continuous, $Dg(t) = V(t)$ for all $t \in \mathbb{R}$. Let us finally define $f = g + \Psi$. Thanks to (27), we deduce that $Df(t) = V(t) + D\Psi(t) \in [-1,1]$ for all $t \in \mathbb{R}$, and therefore, f is 1-Lipschitz. Moreover

$$J_L f(t) = Dg(t) + J_L\Psi(t) = [h_1(t), h_2(t)], \quad \text{for all } t \in \triangle.$$

Using a standard argument, we can modify f outside $[0,1]$ to ensure that it has bounded support. $\square$

4. Smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with surjective derivatives

In this section we show how to construct a $\mathcal{C}^1$-function $g : \mathbb{R}^n \to \mathbb{R}^m$ whose derivative takes in its range all $Q \in \mathbb{R}^{m\times n}$. As before (*c.f.* Theorem 2.1), it will be sufficient to construct a $\mathcal{C}^1$-function g with bounded support such that

$$\mathrm{Im}(Dg) \supset \overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}.$$

We now show how the proof of Theorem 2.1 can be modified to provide an easy proof of the above assertion. Indeed, our set of interest is now the compact metric space $\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}$ (rather than $\mathcal{K}(\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}})$) and Steps 1–2 and 5–6 are now irrelevant, since we deal with singletons.

We proceed as follows: let $h : \Delta \to \overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}$ be a continuous surjection (coding over the Cantor set) and extend it to a continuous (surjective) function

$$\hat{V} : [0,1] \to \overline{\mathrm{B}}_{\mathbb{R}^{m\times n}} \qquad (\text{with } \hat{V}|_{\Delta} = h).$$

Note that, $\hat{V}$ is a Peano-like curve filling the set $\overline{\mathrm{B}}_{\mathbb{R}^{m\times n}}$. We define $\alpha : [0,1] \times \{0\}_{n-1} \to \mathbb{R}^m$ and $\beta : [0,1] \times \{0\}_{n-1} \to \mathbb{R}^{m\times n}$ by (21) and respectively (22) and proceed as in Subsection 3.3 to obtain (by Proposition 2.2) a $\mathcal{C}^1$-function g with bounded support satisfying

$$Dg((t, 0_{n-1})) = \hat{V}(t), \qquad \text{for all } t \in [0,1],$$

which is the desired conclusion.

Remark 4.1. Several authors have worked on questions related to the range of the derivative of a smooth map from $\mathbb{R}^n$ to $\mathbb{R}$ (see *e.g.* [3, 7, 10] and references therein), corresponding to the case $m = 1$. In this case, the question of surjectivity of the derivative map admits a trivial answer (think of the example $g(x) = ||x||^2$ for the Euclidean norm). To the best of our knowledge, there are no results in the literature related to the surjectivity of the derivative map for $m > 1$.

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