Characterizations of evenly convex sets and evenly quasiconvex functions

A. Daniilidis & J.-E. Martinez-Legaz [1]

Abstract The aim of this paper is to present a geometric characterization of even convexity in separable Banach spaces, which is not expressed in terms of dual functionals or separation theorems. As an application, an analytic equivalent definition for the class of evenly quasiconvex functions is derived.

1 Introduction

According to Fenchel [4], a subset of a locally convex real topological vector space X is called evenly convex if it is an intersection of open half-spaces. As a consequence of the Hahn-Banach theorem, every open or closed convex set is evenly convex (note that any closed half-space is an intersection of open half-spaces). Evenly convex sets are precisely those convex sets having the property that for every outside point, there exists a closed hyperplane containing the point and not meeting the set. This interesting property shows the importance of evenly convex sets, and has already appeared in several places in the literature (see [6, Proposition 2], [7], for example). In finite dimensions, evenly convex sets have been recently studied by Rodríguez [14]; as observed there, they are precisely the solution sets of general linear inequality systems (that is, those linear inequality systems in which both strict and nonstrict inequalities may occur). That work contains several interesting characterizations of evenly convex sets, but all of them are in terms of hyperplanes or of exposed faces, so that they are of a separational character. A natural question arises: Is it possible to formulate necessary and sufficient conditions of a geometric nature on a (convex) set ensuring the aforementioned separability from any outside point? Of course, this question is equivalent to asking for a non-separational characterization of evenly convex sets.

Notice that the same question with strong separation has an easy answer: A necessary and sufficient condition for a set to be strongly separated from any outside point by a closed hyperplane is it to be convex and closed. However, the case of evenly convex sets seems to be more complicated: in [4] a non-separational characterization is given, but only for subsets of a finite dimensional Euclidean space. In the first part of this paper, we shall extend this

[1] Research supported by the DGICYT (Spain), Project PB98-0867, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya, Grant 2001SGR-00162. This work was made during a visit of the second author to the University of Pau (June 2001), to which he is grateful for the support received.

The authors wish to thank J. Dutta for having carefully read a preliminary version of this paper.

characterization to the class of separable Banach spaces. We shall also present an example showing that the same characterization is not valid in more general spaces.

The second part of the paper is devoted to the study of evenly quasiconvex functions [11] (called normal quasiconvex functions in [8]), that is, extended real-valued functions $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ whose level sets are evenly convex. Obviously, every lower semicontinuous quasiconvex function is evenly quasiconvex. It is well-known and easy to prove that all upper semicontinuous quasiconvex functions are evenly quasiconvex, too. Evenly quasiconvex functions are important in duality theory. In fact, as lower semicontinuous proper convex functions constitute the regular subclass of convex functions under Fenchel conjugation, it was early recognized that evenly quasiconvex functions are the regular functions in all usual quasiconvex conjugation schemes (see [8], [11], [9], [13], for example). In Mathematical Economics, evenly quasiconvex functions also play a role. More specifically, the indirect utility functions arising in consumer theory are characterized as the non-increasing evenly quasiconvex functions that satisfy an additional minor regularity condition [10]. But despite the importance of evenly quasiconvex functions, no analytical characterization is known so far. In other words, the only available method for checking the even quasiconvexity of a function that is neither lower nor upper semicontinuous is looking whether each of its level sets enjoys the separation property that characterizes evenly convex sets. In this paper we give such an analytic characterization of evenly quasiconvex functions in separable Banach spaces, which follows from our geometric characterization of evenly convex sets.

Throughout the rest of this paper, X denotes a Banach space, with topological dual X^*. By $\langle p, x\rangle$ we denote the dual coupling for any $x \in X$ and $p \in X^*$. A nonempty subset P of X is called a cone, provided $\lambda P \subset P$, for all $\lambda \geq 0$. A cone P is convex if and only if $P + P \subset P$. If P is a convex cone, then the set $\ell(P) := P \cap (-P)$ is the largest linear subspace of X contained in P (see [5]). The convex cone P is said to be pointed, provided $\ell(P) = \{0\}$. Given a nonempty convex subset K of X and a point x_0 in $\mathrm{cl}(K)$, the (Bouligand) tangent cone of K at x_0 is the closed convex cone $T_K(x_0) := \overline{\bigcup_{\lambda>0} \lambda\,(K - \{x_0\})}$. Finally, let $\mathrm{co}(S)$ denote the convex hull of a subset S of X.

2 Evenly convex sets

Let X be a Banach space and K be a nonempty convex subset of X.

Definition 1 *A set K is said to be evenly convex, if for every $x_0 \in X \setminus K$, there exists $q \in X^*$ such that $\langle q, x - x_0\rangle < 0$, for all $x \in K$.*

According to this definition, a set K is evenly convex if, and only if, it can be written as an intersection of open half spaces. This shows that intersections of evenly convex sets are evenly convex; consequently, since the whole space X is evenly convex (being the intersection of the empty family), the evenly convex hull $\operatorname{eco}(S)$ of an arbitrary set $S \subset X$ is well defined as the intersection of all evenly convex sets containing S. An easy application of Hahn-Banach theorem shows that any closed or open convex set is evenly convex. It is also straightforward that every convex subset of $\mathbb{R}$ is evenly convex. However, this is not the case if $\dim X \geq 2$, as is easily seen by considering the union of an open half-space with one of its boundary points. One can also observe that K is evenly convex if, and only if, for every $x_0 \in \operatorname{cl}(K) \setminus K$ the set $K \cup \{x_0\}$ is convex and has x_0 as an exposed point.

The following proposition will be needed in the sequel.

Proposition 2 *Let P be a closed convex cone in a separable Banach space X. Then there exists $p \in X^*$ such that $\langle p, x\rangle > 0$ for every $x \in P \setminus \ell(P)$ and $\langle p, x\rangle = 0$ for every $x \in \ell(P)$.*

Proof Clearly, $\ell(P) := P \cap (-P)$ is a closed linear subspace of X. Let us consider the quotient space $Z = X/\ell(P)$ and the canonical projection $\pi : X \to Z$ (that is, $\pi(x) = x + \ell(P)$, for every x in X). Then $\pi(P)$ is a closed convex pointed cone of Z. Indeed, one can easily check that $\pi(P)$ is a convex pointed cone, while its closedness follows from the closedness of P in X and the definition of the quotient topology, since $\pi^{-1}(\pi(P)) = P + \ell(P) = P$.

Let us further consider the w^*-closed set

$$\pi(P)^* := \{z^* \in Z^* : \langle z^*, x\rangle \geq 0, \text{ for all } x \in \pi(P)\}. \tag{1}$$

Since Z is a separable Banach space, using [1, Theorem 2.19] we conclude that the set w^*-qri $[\pi(P)^*]$ of the w^*-quasi-relative interior points of $\pi(P)^*$ (see [1, Definition 2.3]) is nonempty. If $\bar{z}^* \in Z^*$ is any such point, then we have $\langle \bar{z}^*, z\rangle > 0$, for all $z \in \pi(P) \setminus \{0\}$. Indeed, if for some $\bar{z} \in \pi(P) \setminus \{0\}$ we had $\langle \bar{z}^*, \bar{z}\rangle = 0$, then using (1) we would obtain $\langle \bar{z}^* - z^*, \bar{z}\rangle \leq 0$ for all $z^* \in \pi(P)^*$, which contradicts [1, Proposition 2.16].

Let now $p = \pi^*(\bar{z}^*)$, where $\pi^* : Z^* \to X^*$ is the adjoint operator of π. It follows easily that $\langle p, x\rangle > 0$ for every $x \in P \setminus \ell(P)$ and $\langle p, x\rangle = 0$ for every $x \in \ell(P)$. □

The main result in this section is based on the following proposition:

Proposition 3 *Let K be a convex subset of a Banach space X and let $x_0 \in \operatorname{cl}(K) \setminus K$. We consider the following assertions:*

(i) $x_0 \notin \operatorname{eco}(K)$.

(ii) $\exists q \in X^* : \langle q, x - x_0\rangle < 0$, *for all x in K.*

(iii) $[x_0 + \ell(T_K(x_0))] \cap K = \emptyset$.

Then (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii). Moreover, if X is separable, then all three assertions are equivalent.

Proof The equivalence (i)$\Longleftrightarrow$(ii) follows directly from Definition 1 and the observation that $\mathrm{eco}(K)$ is the intersection of all open half-spaces that contain the set K.

Let us prove (ii)$\Longrightarrow$(iii). Assertion (ii) implies $\langle q, d\rangle \leq 0$, for all $d \in T_K(x_0)$. If (iii) does not hold, then for some $x \in K$ we would have $x - x_0 \in T_K(x_0) \cap (-T_K(x_0))$, that is $\langle q, x - x_0\rangle = 0$, which is a contradiction.

Finally, let us assume that (iii) holds, and that X is a separable Banach space. Then let us set $P = T_K(x_0)$. It follows that P is a closed convex cone. Hence, there exists $q \in X^*$ such that $\langle q, x\rangle < 0$ for every $x \in P \setminus \ell(P)$ and $\langle q, x\rangle = 0$ for every $x \in \ell(P)$ (take $q = -p$ in Proposition 2). Let now $x \in K$. Since $x - x_0 \in P$, we get $\langle q, x - x_0\rangle \leq 0$. In fact, this inequality must be strict. Indeed, let us assume that $\langle q, x - x_0\rangle = 0$. Then $x - x_0 \in \ell(P)$, that is $x \in x_0 + \ell(P)$, which obviously contradicts (iii). Consequently $\langle q, x - x_0\rangle < 0$, for all x in K, that is (ii) (or equivalently (i)) holds. □

Remark: The equivalence between (ii) and (iii) had essentially been given also in [4] for the case of finite dimensional spaces. Proposition 2 plays here a crucial role for the extension to separable Banach spaces.

Using the above proposition, we obtain the following formula for the evenly convex hull of a set.

Corollary 4 *Let K be any subset of a separable Banach space X. Then*

$$\mathrm{eco}(K) = \{x \in \mathrm{cl}(\mathrm{co}(K)) : \; [x + \ell(T_{co(K)}(x))] \cap \mathrm{co}(K) \neq \emptyset\}.$$

Proof Set $K_1 = \{x \in \mathrm{cl}(\mathrm{co}(K)) : [x + \ell(T_{\mathrm{co}(K)}(x))] \cap \mathrm{co}(K) \neq \emptyset\}$ and note that $\mathrm{co}(K) \subset K_1$. Let now $x \in \mathrm{eco}(K) \setminus \mathrm{co}(K)$. Then obviously $x \in \mathrm{cl}(\mathrm{co}(K)) \setminus \mathrm{co}(K)$, hence by Proposition 3(iii)$\Longrightarrow$(i) we get $[x + \ell(T_{\mathrm{co}(K)}(x))] \cap \mathrm{co}(K) \neq \emptyset$. Consequently, $x \in K_1$, hence $\mathrm{eco}(K) \subset K_1$.

Let now $x \in K_1$. Again, if $x \in \mathrm{co}(K)$, then obviously $x \in \mathrm{eco}(K)$. If $x \notin \mathrm{co}(K)$, then Proposition 3(i)$\Longrightarrow$(iii) shows that $x \in \mathrm{eco}(\mathrm{co}(K)) = \mathrm{eco}(K)$. □

We now obtain immediately the following characterization of even convexity, which does not make explicit use of dual functionals.

Theorem 5 *Let K be a convex subset of a separable Banach space X. Then the following are equivalent:*

(i) K is evenly convex.

(ii) $[x_0 + \ell(T_K(x_0))] \cap K = \emptyset$, $\forall x_0 \in \mathrm{cl}(K) \setminus K$.

Proof Since K is convex, we have $K = \mathrm{co}(K)$. Hence, by Corollary 4, K is evenly convex if, and only if, $K = \{x \in \mathrm{cl}(K) : [x + \ell(T_K(x))] \cap K \neq \emptyset\}$. But this condition is obviously equivalent to (ii). □

Remark:

1. Condition (ii) states that if a point $x_0 \in \mathrm{cl}(K)$ does not belong to K, then every line of its tangent cone that passes through 0 does not intersect $K - \{x_0\}$. Note that this is trivially satisfied if $T_K(x_0)$ is pointed.

2. Implication (i)$\Longrightarrow$(ii) in Theorem 5 is valid in arbitrary Banach spaces. On the other hand, separability assumption is crucial for (ii)$\Longrightarrow$(i) as shows the following example:

Example: Let I be any uncountable set and consider the non-separable Hilbert space

$$X = \ell^2(I) := \left\{ x = (x_i)_i \in \mathbb{R}^I \; : \; \sup_{\substack{F \subset I \\ F \text{ finite}}} \sum_{k \in F} |x_{i_k}|^2 < +\infty \right\}, \tag{2}$$

that is, the space of all square summable functions $x : I \to \mathbb{R}$. Then, for the convex set $K = \ell^2(I)_+ \setminus \{0\}$ it is easily seen that $\mathrm{cl}(K) \setminus K = \{0\}$ and $T_K(0) = \ell^2(I)_+$. Since the latter is a (closed convex) pointed cone, condition (ii) is satisfied. However, there is no functional $q = (q_i) \in X^* = \ell^2(I)$ such that $\langle q, x \rangle < 0$ for every $x \in K$. Indeed, if such a functional exists, then it should satisfy $q_i < 0$ for all $i \in I$, which is impossible, since (2) implies that the set $\{i \in I : q_i \neq 0\}$ is countable. (See also [1, Example 3.11 (iii)] for an example of the same type.)

Corollary 6 *A convex set K in a separable Banach space is evenly convex if, and only if, for every $x_0 \in \mathrm{cl}(K) \setminus K$, $\{x_n\}_{n \geq 1} \subset K$, $\{\lambda_n\}_{n \geq 1} \subset (0, +\infty)$ and $d \in X$ such that $\lim_{n \to \infty} \lambda_n (x_n - x_0) = d$ one has $x_0 - d \notin K$.*

Proof Assume first that K is evenly convex. Let us note that if d is as above, then $d \in T_K(x_0)$. If $x_0 - d \in K$, then obviously $-d \in T_K(x_0)$, whence $-d \in \ell(T_K(x_0))$, which violates condition (ii) of Theorem 5, since $x_0 - d \in [x_0 + \ell(T_K(x_0))] \cap K$. Conversely, if K is not evenly convex then that condition is violated, that is, there exists $d' \in T_K(x_0)$ such that $K \cap \{x_0 - td'; t > 0\} \neq \emptyset$. Then for some $t > 0$, and for $d = td'$ we would have $x_0 - d \in K$, which contradicts the assumption in the corollary. □

3 Evenly quasiconvex functions

We recall that a function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is quasiconvex (respectively, lower semicontinuous), if for every $a \in \mathbb{R}$ the level set $S_f(a) := \{x \in X : f(x) \leq a\}$ is convex (respectively, closed). The class of lower semicontinuous quasiconvex functions has an important role in optimization ([2], [3] e.g.) and presents good stability properties (see [12], for example). It turns out that the larger class of evenly quasiconvex functions (see definition below) appears naturally in the quasiconvex duality ([8], [11], [9], [13]) and enjoys applications in economics [10].

Definition 7 *A function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called evenly quasiconvex, if for every $a \in \mathbb{R}$ the level set $S_f(a) := \{x \in X : f(x) \leq a\}$ is evenly convex.*

It is obvious that every lower semicontinuous quasiconvex function is evenly quasiconvex. The class of evenly quasiconvex functions is strictly larger, since it also contains all upper semicontinuous quasiconvex functions (that is, functions whose strict level sets are convex and open). Considering indicator functions of appropriate convex sets one can get examples of evenly quasiconvex functions that are neither lower semicontinuous nor upper semicontinuous as well as quasiconvex functions that are not evenly quasiconvex. The following example shows that an evenly quasiconvex function might fail to have evenly convex strict level sets.

Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} 0, & \text{if } x \geq y \text{ and } y \leq 0, \\ y/x, & \text{if } x > y > 0, \\ 1, & \text{elsewhere.} \end{cases}$$

Then f has closed convex level sets, hence in particular it is evenly quasiconvex; however the strict level set $S_f^-(1) := \{(x,y) \in \mathbb{R}^2 \colon\ x > y > 0\} \cup \{(x,y) \in \mathbb{R}^2 \colon x \geq y \text{ and } y \leq 0\}$ is not evenly convex.

Evenly quasiconvex functions are closed under pointwise suprema, given that the level sets of the pointwise supremum of a family of functions are intersections of level sets of the members of the family. Therefore every function has a largest evenly quasiconvex minorant, which is called its evenly quasiconvex hull and denoted by f_{eq}. The following expression for f_{eq} is well-known ([15, pg 144] e.g.); we include a proof in order to make the paper self-contained.

Proposition 8 *For any $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $x \in X$, one has*

$$f_{eq}(x) = \inf \{a \in \mathbb{R} : x \in \operatorname{eco}(S_f(a))\}. \tag{3}$$

Proof Since $f_{eq} \leq f$ and the level sets of f_{eq} are evenly convex, one has $\operatorname{eco}(S_f(a)) \subset S_{f_{eq}}(a)$ for every $a \in \mathbb{R}$. Consequently, it follows easily that $f_{eq}(x) = \inf \left\{a \in \mathbb{R} : x \in S_{f_{eq}}(a)\right\} \leq \inf \{a \in \mathbb{R} : x \in \operatorname{eco}(S_f(a))\}$.
To prove the opposite inequality, we set $g(x) = \inf \{a \in \mathbb{R} : x \in \operatorname{eco}(S_f(a))\}$. In view of the definition of f_{eq} it suffices to show that g is an evenly quasiconvex minorant of f. But this easily follows from the inequality $g(x) = \inf \{a \in \mathbb{R} : x \in \operatorname{eco}(S_f(a)\} \leq \inf \{a \in \mathbb{R} : x \in S_f(a)\} = f(x)$ and the fact that, for every $a \in \mathbb{R}$, one has $S_g(a) = \bigcap_{b>a} \operatorname{eco}(S_f(b))$. $\square$

Definition 9 *[15, Definition 4.3] A function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called evenly quasiconvex at $x_0 \in X$, if $f_{eq}(x_0) = f(x_0)$.*

Since the evenly quasiconvex hull of a function lies between its lower semicontinuous quasiconvex hull and its quasiconvex hull, it follows that if a quasiconvex function f is lower semicontinuous at x_0 then it is also evenly quasiconvex at x_0.

The following characterization of even quasiconvexity at a point will be useful:

Proposition 10 *A function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is evenly quasiconvex at $x_0 \in X$ if and only if for all $a < f(x_0)$ one has $x_0 \notin \mathrm{eco}(S_f(a))$.*

Proof If f is evenly quasiconvex at x_0 and $a < f(x_0)$ then $a < f_{eq}(x_0)$, that is, $x_0 \notin S_{f_{eq}}(a)$, whence, as $\mathrm{eco}(S_f(a)) \subset S_{f_{eq}}(a)$, $x_0 \notin \mathrm{eco}(S_f(a))$. Conversely, if $x_0 \notin \mathrm{eco}(S_f(a))$ for all $a < f(x_0)$ then, by formula (3), one has $f_{eq}(x_0) \geq f(x_0)$; since $f_{eq} \leq f$, it follows that f is evenly quasiconvex at x_0. □

Proposition 11 *Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be any function. Then*
(i) f is evenly quasiconvex at $x_0 \in X$ if, and only if, for every $a < f(x_0)$ there exists $q \in X^*$ such that

$$\langle q, x - x_0 \rangle < 0, \ \forall x \in S_f(a).$$

(ii) f is evenly quasiconvex if, and only if, it is evenly quasiconvex at every $x_0 \in X$.

Proof (i) It follows directly combining Proposition 10 with Proposition 3(i)$\Longleftrightarrow$(ii).
(ii) It is an immediate consequence of the obvious fact that f is evenly quasiconvex if, and only if, $f_{eq} = f$. □

Based on Proposition 3 we obtain the following characterization of even quasiconvexity in separable Banach spaces.

Proposition 12 *Let X be a separable Banach space, $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ a quasiconvex function and $x_0 \in X$. Then f is evenly quasiconvex at x_0 if, and only if, the following condition holds:*

(*) for every $y_0 \in X$ such that $f(y_0) < f(x_0)$, every $\{y_n\}_{n \geq 1} \subset X$ such that $\lim_{n \to +\infty} y_n = y_0$ and every $\{\mu_n\}_{n \geq 1} \subset (0, +\infty)$, one has

$$f(x_0) \leq \liminf_{n \to +\infty} f(x_0 + \mu_n(x_0 - y_n)).$$

Proof Let us first assume that f is evenly quasiconvex at x_0. If (*) is not satisfied, then there exists $y_0 \in X$ with $f(y_0) < f(x_0)$, $\{y_n\}_{n \geq 1} \subset X$, $\{\mu_n\}_{n \geq 1} \subset (0, +\infty)$, such that $\{y_n\} \to y_0$ and $\lim_{n \to +\infty} f(x_0 + \mu_n(x_0 - y_n)) < f(x_0)$. Set $\beta = \max\{f(y_0), \lim_{n \to +\infty} f(x_0 + \mu_n(x_0 - y_n))\}$. Let $\beta < a < f(x_0)$. By Proposition 11 there exists $q \in X^*$ such that $\langle q, x_0 \rangle > \langle q, x \rangle$, for all $x \in S_f(a)$. In particular, for sufficiently large n we have $\langle q, x_0 \rangle > \langle q, x_0 + \mu_n(x_0 - y_n) \rangle$, yielding $\langle q, y_n \rangle > \langle q, x_0 \rangle$, and consequently $\langle q, y_0 \rangle \geq \langle q, x_0 \rangle$. This contradicts the fact that $y_0 \in S_f(a)$.

It remains to show that condition (*) implies that f is evenly quasiconvex at x_0. Suppose, towards a contradiction, that f is not evenly quasiconvex at x_0. Then by Proposition 10, there exists $a < f(x_0)$ such that $x_0 \in \operatorname{eco}(S_f(a))$. Then obviously $x_0 \in \operatorname{cl}(S_f(a)) \setminus (S_f(a))$. Then by Proposition 3(iii)$\Longrightarrow$(i) we would have $[x_0 + \ell(T_{S_f(a)}(x_0))] \cap S_f(a) \neq \emptyset$. Hence there exists $y_0 \in S_f(a)$ such that $x_0 - y_0 \in T_{S_f(a)}(x_0)$. This means that $x_0 - y_0 = \lim_{n \to +\infty} \lambda_n(z_n - x_0)$, for some $\{\lambda_n\}_{n \geq 1} \subset (0, +\infty)$, $\{z_n\}_{n \geq 1} \subset S_f(a)$. Let us define for every $n \geq 1$, $y_n = x_0 + \lambda_n(x_0 - z_n)$ and $\mu_n = 1/\lambda_n$. Then it follows that $\lim_{n \to +\infty} y_n = y_0$ and $\liminf_{n \to +\infty} f(x_0 + \mu_n(x_0 - y_n)) = \liminf_{n \to +\infty} f(z_n) \leq a$, since $\{z_n\}_{n \geq 1} \subset S_f(a)$. This clearly contradicts (*), since $a < f(x_0)$. This finishes the proof. □

References

[1] J. Borwein & A. Lewis, Partially finite convex programming, Part I: Quasi relative interiors and duality theory, *Math. Program.* **57** (1992), 15–48.

[2] J.-P. Crouzeix, *Contribution à l'étude des fonctions quasiconvexes*, Thesis, Université de Clermont, 231 pp., 1977.

[3] J.-P. Crouzeix, *Continuity and differentiability properties of quasiconvex functions on* $\mathbb{R}^n$, in: Generalized Concavity in Optimization and Economics, S. Schaible and W. Ziemba (eds.), Academic Press, New York (1981), 109-130.

[4] W. Fenchel, A remark on convex sets and polarity, *Comm. Sem. Math. Univ. Lund (Medd. Lunds Univ. Math. Sem.) Tome Supplém.* (1952), 82-89.

[5] D. T. Luc, *Theory of vector optimization*, Lecture Notes in Economics and Mathematical Systems **319**, Springer, Berlin (1988).

[6] N. Hadjisavvas, D. Kravvaritis, G. Pantelidis & I. Polyrakis, Nonlinear monotone operators with values in $L(X, Y)$, *J. Math. Anal. Appl.* **140** (1989), 83-94.

[7] N. Hadjisavvas, D. Kravvaritis, G. Pantelidis & I. Polyrakis, Hereditary Order Convexity in $L(X, Y)$, *Rend. Circ. Mat. Palermo* **38** (1989), 130–139.

[8] J.-E. Martinez-Legaz, *A generalized concept of conjugation*, in: Optimization, theory and algorithms, J.-B. Hiriart-Urruty, W. Oettli and J. Stoer (eds.), Lecture Notes in Pure and Applied Mathematics **86** (1983), 45-59, Marcel Dekker, New York.

[9] J.-E. Martinez-Legaz, Quasiconvex duality theory by generalized conjugation methods, *Optimization* **19** (1988), 603-6 52.

[10] J.-E. MARTINEZ-LEGAZ, Duality between direct and indirect utility functions under minimal hypotheses, *J. Math. Econ.* **20** (1991), 199-209.

[11] U. PASSY & E. Z. PRISMAN, Conjugacy in quasi-convex programming, *Math. Program.* **30** (1984), 121-146.

[12] J.-P. PENOT, *Generalized convexity in the light of nonsmooth analysis*, in: Recent Developments in Optimization, R. Durier and C. Michelot (eds.), Lecture Notes in Economics and Math. Systems **429** (1995), 269-290, Springer Verlag, Berlin.

[13] J.-P. PENOT & M. VOLLE, On quasi-convex duality, *Math. Oper. Res.* **15** (1990), 597-625.

[14] M. RODRIGUEZ, *Nuevos resultados sobre sistemas lineales y conjuntos convexos*, Thesis, Universidad de Alicante, 126 pp., 2001.

[15] I. SINGER, *Abstract Convex Analysis*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1997.

Aris Daniilidis

Laboratoire de Mathématiques Appliquées
Université de Pau et des Pays de l'Adour
Avenue de l'Université
64000 Pau, France

Current address:
INRIA, Rhône-Alpes, Monbonnot, 38334 St Ismier Cedex, France

E-mail: `aris.daniilidis@inrialpes.fr`

Juan-Enrique Martínez-Legaz

CODE and Departament d'Economia i d'Història Economica
Universitat Autònoma de Barcelona
Bellaterra 08193, Spain

E-mail: `jemartinez@selene.uab.es`