Subdifferential representation of convex functions: refinements and applications

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Abstract Every lower semicontinuous convex function can be represented through its subdifferential by means of an "integration" formula introduced in [10] by Rockafellar. We show that in Banach spaces with the Radon-Nikodym property this formula can be significantly refined under a standard coercivity assumption. This yields an interesting application to the convexification of lower semicontinuous functions.

Key words Convex function, subdifferential, epi-pointed function, cusco mapping, strongly exposed point.

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1 Introduction

Let X be a Banach space and $g : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous convex function. Rockafellar [10] has shown that g can be represented through its subdifferential ∂g as follows:

$$g(x) = g(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \; + \; \langle x_n^*, x - x_n \rangle \right\}, \tag{1}$$

for every $x \in X$, where x_0 is an arbitrary point in the domain of ∂g and where the above supremum is taken over all integers n, all $x_1, ..., x_n$ in X and all $x_0^* \in \partial g(x_0), x_1^* \in \partial g(x_1), \ldots, x_n^* \in \partial g(x_n)$ (for $n = 0$ we take the convention $\sum_{i=0}^{-1} = 0$). In this paper we show that, in Banach spaces with the Radon-Nikodym property (Definition 2), and under a standard coercivity assumption on g, formula (1) can be considerably simplified. Namely, it suffices to estimate the above supremum among the set of strongly exposed points of g (Definition 12), instead of the much larger set of all points of the domain of ∂g.

This simple geometrical fact has also the following consequence: the closed convex envelope of a non-convex function f satisfying the same coercivity condition can be recovered by the Fenchel subdifferential ∂f of f through formula (1), and this despite the fact that for non-convex functions, this subdifferential may be empty at many points. This last result generalizes the ones obtained in [1, Proposition 2.7], [2, Theorem 3.5] in finite dimensions.

2 Preliminaries

Throughout the paper we denote by X a Banach space and by X^* its dual space. In the sequel, we denote by $\hat{\imath} : X \rightsquigarrow X^{**}$ the isometric embedding of X into its second dual space X^{**}. Given $x \in X$, $x^* \in X^*$ and $x^{**} \in X^{**}$, we denote by $\langle x^*, x \rangle$ (respectively, $\langle x^*, x^{**} \rangle$) the value of the functional x^* at x (respectively, the value of x^{**} at x^*). Note also that with this notation we have $\langle x^*, \hat{\imath}(x) \rangle = \langle x^*, x \rangle$. For $x \in X$ and $\rho > 0$ we denote by $B(x, \rho)$ the open ball centered at x with radius ρ.

If $f : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real valued function, we denote by

$$\text{epi} f = \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\}$$

its epigraph, and by

$$\text{dom}\, f := \{x \in X : f(x) \in \mathbb{R}\}$$

its domain. When the domain of f is nonempty we say that f is proper. By the term subdifferential, we always mean the Fenchel subdifferential ∂f defined for every $x \in \operatorname{dom} f$ as follows

$$\partial f(x) = \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle, \forall y \in X\}.$$

If $x \in X \setminus \operatorname{dom} f$, we set $\partial f(x) = \emptyset$. The domain of the subdifferential of f is defined by

$$\operatorname{dom} \partial f = \{x^* \in X^* : \partial f(x) \neq \emptyset\}.$$

For a proper lower semicontinuous function f, its closed convex envelope $\overline{\text{co}} f : X \to \mathbb{R} \cup \{+\infty\}$ can be defined through its epigraph via the formula

$$\operatorname{epi}(\overline{\text{co}} f) = \overline{\text{co}}(\operatorname{epi} f),$$

where $\overline{\text{co}}(\operatorname{epi} f)$ is the closed convex hull of $\operatorname{epi} f$ in the Banach space $X \times \mathbb{R}$ endowed with the norm $(x, t) \mapsto \left(\|x\|^2 + |t|^2\right)^{1/2}$ for all $(x, t) \in X \times \mathbb{R}$. If $f^{**} : X^{**} \longrightarrow \mathbb{R} \cup \{+\infty\}$ denotes the Legendre-Fenchel biconjugate of f, then it is well-known that $\overline{\text{co}} f = f^{**} \circ \hat{\imath}$, that is, for every $x \in X$

$$(\overline{\text{co}} f)(x) = f^{**}(\hat{\imath}(x)) = \sup_{x^* \in X^*} \{\langle x^*, x\rangle - f^*(x^*)\},$$

where $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is the Legendre-Fenchel conjugate of f, that is the proper lower semicontinuous convex function defined for all $x^* \in X^*$ by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

Note also that for any $x \in X$ and $x^* \in X^*$ we have:

$$x^* \in \partial f(x) \Longleftrightarrow \hat{\imath}(x) \in \partial(f^*)(x^*). \tag{2}$$

Let C be a non-empty closed convex subset of X. We denote by $\sigma_C : X^* \to \mathbb{R} \cup \{+\infty\}$ the Legendre-Fenchel conjugate of the indicator function of C, that is, for all $p \in X^*$

$$\sigma_C(p) = \sup_{u \in C} \langle p, u\rangle.$$

Note that σ_C is a positively homogeneous convex function. Its relationship with the Legendre-Fenchel conjugate of a proper lower semicontinuous convex function g is as follows:

$$t g^*(t^{-1} x^*) = \sigma_{\operatorname{epi} g}(x^*, -t),$$

for all $t > 0$ and all $x^* \in X^*$. In particular, using the fact that $\operatorname{dom} \sigma_{\operatorname{epi} g}$ and $\operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} g})$ are convex cones, it is easily seen that

$$x^* \in \operatorname{int}(\operatorname{dom} g^*) \Longleftrightarrow (x^*, -1) \in \operatorname{int}(\operatorname{dom} \sigma_{\operatorname{epi} g}). \tag{3}$$

Finally, we denote by $N_C(u)$ the set of normal directions of C at a point $u \in C$, that is,

$$N_C(u) = \{p \in X^* : \langle p, v - u\rangle \leq 0, \quad \forall v \in C\}.$$

Its relationship with the subdifferential of a proper lower semicontinuous convex function g is as follows

$$t^{-1} x^* \in \partial g(x) \Longleftrightarrow (x^*, -t) \in N_{\operatorname{epi} g}(x, g(x)), \tag{4}$$

where $t > 0$, $x \in X$ and $x^* \in X^*$.

2.1 Strongly exposed points and Radon-Nikodym property

Let us recall from [9, Definition 5.8] the following definition.

Definition 1 *Let C be a non-empty closed convex subset of X. A point $u \in C$ is said strongly exposed if there exists $p \in X^*$ such that for each sequence $\{u_n\} \subset C$ the following implication holds*

$$\lim_{n\to+\infty} \langle p, u_n\rangle = \sigma_C(p) \implies \lim_{n\to+\infty} u_n = u.$$

In such a case $p \in X^*$ is said to be a "strongly exposing" functional for the point u in C. We denote by $\mathrm{Exp}\,(C,u)$ the set of all functionals of X^* satisfying this property.

Let us further denote by $\exp C$ the set of strongly exposed points of C. Clearly, $u \in \exp C$ if, and only if, $\mathrm{Exp}(C,u) \neq \emptyset$. It follows directly that for every $u \in C$ we have the inclusion

$$\mathrm{Exp}(C,u) \subset N_C(u) \cap \mathrm{dom}\,\sigma_C. \tag{5}$$

We also denote by Exp C the set of all strongly exposing functionals, that is,

$$\mathrm{Exp}\ C = \bigcup_{u\in\exp C} \mathrm{Exp}(C,u).$$

We also recall (see [9, Theorem 5.21], for example) the following definition.

Definition 2 *A Banach space X is said to have the Radon-Nikodym property, if every non-empty closed convex bounded subset C of X can be represented as the closed convex hull of its strongly exposed points, that is,*

$$C = \overline{\mathrm{co}}(\exp C).$$

Examples of Radon-Nikodym spaces are reflexive Banach spaces and separable dual spaces.

Let us mention that, in spaces with the Radon-Nikodym property, the set $\mathrm{Exp}\,C$ of strongly exposing functionals of a nonempty closed convex bounded set C is dense in X^*. Moreover, the boundedness of C implies that $\mathrm{dom}\,\sigma_C = X^*$. In case of unbounded sets, one has the following result.

Proposition 3 *Suppose that X has the Radon-Nikodym property and C is a nonempty closed convex set. Then $\mathrm{Exp}\,C$ is dense in $\mathrm{int}(\mathrm{dom}\,\sigma_C)$.*

Proof If $\mathrm{int}(\mathrm{dom}\,\sigma_C) = \emptyset$ the assertion holds trivially. Let us assume that $U := \mathrm{int}(\mathrm{dom}\,\sigma_C) \neq \emptyset$ and let us note that the w*-lower semicontinuous convex function σ_C is continuous on the open set U, see [9, Proposition 3.3]. Using Collier's characterization of the Radon-Nikodym property ([7, Theorem 1]), we conclude that σ_C is Fréchet differentiable in a dense subset D of U. For every $p_0 \in D$, Smulian's duality guarantees that there exists $u_0 \in \exp C$ such that $u_0 = \nabla^F \sigma_C(p_0)$ (see [9], for example). In particular, $p_0 \in \mathrm{Exp}(C,u_0)$, hence $p_0 \in \mathrm{Exp}\,C$. The proof is complete. $\square$

2.2 Cyclically monotone operators

Given a set-valued operator $T : X \rightrightarrows X^*$, we denote its domain by $\mathrm{dom}\,T = \{x \in X : T(x) \neq \emptyset\}$, its image by

$$\mathrm{Im}\,T = \bigcup_{x\in X} T(x)$$

and its graph by

$$\mathrm{Gr}\,T := \{(x,x^*) \in X \times X^* :\ x^* \in T(x)\}.$$

We also denote by $T^{-1} : X^* \rightrightarrows X$ the inverse operator, defined for every $(x,x^*) \in X \times X^*$ by the relation

$$x \in T^{-1}(x^*) \Longleftrightarrow x^* \in T(x).$$

Clearly $\operatorname{dom} T^{-1} = \operatorname{Im} T$.

The operator T is called *cyclically monotone* (respectively, *monotone*) if for all integers $n \geq 1$ (respectively, for $n = 2$), all $x_1, \ldots, x_n$ in X and all $x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ we have

$$\sum_{i=1}^{n} (x_i^*, x_{i+1} - x_i) \leq 0,$$

where $x_{n+1} := x_1$. It is called maximal cyclically monotone (respectively, maximal monotone), if its graph cannot be strictly contained in the graph of any other cyclically monotone (respectively, monotone) operator.

We recall from [10] (see also [9]) the following fundamental results:

Proposition 4 *The subdifferential ∂g of a proper lower semicontinuous convex function g is both a maximal monotone and a maximal cyclically monotone operator.*

Proposition 5 *Let T be a cyclically monotone operator and let $x_0 \in \operatorname{dom} T$. Consider the function $h : X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$h(x) := \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \ + \ \langle x_n^*, x - x_n \rangle \right\}, \tag{6}$$

where the supremum is taken for all integers n, all $x_1, ..., x_n$ in $\operatorname{dom} T$ and all $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$. Then h is a proper lower semicontinuous convex function and

$$\operatorname{Gr} T \subset \operatorname{Gr} \partial h.$$

We shall refer to (6) by the term "Rockafellar integration formula". The following lemma will be very useful in the sequel. Let us recall that the operator T^{-1} is said to be locally bounded on a non-empty open subset V of X^*, provided that for every $x^* \in V$ there exist $\rho > 0$ such that $T^{-1}(B(x^*, \rho))$ is bounded.

Lemma 6 *Let V be a non-empty open subset of X^*. With the notation of Proposition 5, let us suppose that $\operatorname{Im} T$ is dense in V and T^{-1} is locally bounded on V. Then we have the inclusion*

$$V \subset \operatorname{int}(\operatorname{dom} h^*),$$

where the function h is defined by relation (6) and h^* is its conjugate function.

Proof Fix any $x_0^* \in T(x_0)$. Let $x^* \in V$. Since T^{-1} is locally bounded on V, there exist $\rho > 0$ and $M > 0$ such that $T^{-1}(B(x^*, \rho)) \subset B(0, M)$. Moreover we can suppose that $B(x^*, \rho) \subset V$ since V is an open subset.

Let now $z^* \in B(x^*, \rho) \cap \operatorname{Im} T$. There exists $z \in X$ such that $z^* \in T(z)$. Then formula (6) implies that for all $x \in X$

$$h(x) \geq \langle x_0^*, z - x_0 \rangle \ + \ \langle z^*, x - z \rangle.$$

Using the definition of the conjugate function we obtain

$$h^*(z^*) \leq \langle x_0^*, x_0 \rangle \ + \ \langle z^* - x_0^*, z \rangle \leq M_1.$$

where $M_1 := \|x_0^*\|.\|x_0\| + (\|x^* - x_0^*\| + \rho)\, M$. Hence we have proven that

$$h^* \leq M_1$$

on $B(x^*, \rho) \cap \operatorname{Im} T$.

Since $\operatorname{Im} T$ is dense in V and h^* is lower semicontinuous, this last inequality remains true on $B(x^*, \rho)$. Thus $x^* \in \operatorname{int}(\operatorname{dom} h^*)$. □

2.3 w*-cusco and minimal w*-cusco mappings

Let $T : X \rightrightarrows X^*$ be a set-valued operator. T is said to be w*-upper semicontinuous at $x \in X$, if for every w*-open set W containing $T(x)$ there exists $\rho > 0$ such that $T(B(x,\rho)) \subset W$.

We recall from [4] (see also [5]) the following definition.

Definition 7 *Let U be an open subset of X. T is said to be w*-*cusco *on U, if it is w*-upper semicontinuous with nonempty w*-compact convex values at each point of U. It is said to be* minimal *w**-cusco *on U if its graph does not strictly contain the graph of any other w*-cusco mapping on U.*

In the sequel, we shall need the following result (see [5, Theorem 2.23]).

Proposition 8 *Let U be an open set of X such that $U \subset \operatorname{dom} T$. If T is maximal monotone then it is also minimal w*-cusco on U.*

Further, given a set-valued operator $S : X \rightrightarrows X^*$ we can consider w*-cusco mappings T that are minimal under the property of containing the graph of S. We recall from [5, Proposition 2.3] the following "uniqueness" result that will be in use in the sequel.

Proposition 9 *Let U be an open set of X such that $\operatorname{dom} S$ is dense in U. If the graph of S is contained in the graph of some w*-cusco mapping on U, then there exists a unique w*-cusco mapping on U that contains the graph of S and that is minimal under this property.*

3 Refined representations of convex functions

Throughout this section $g : X \to \mathbb{R} \cup \{+\infty\}$ will denote a proper lower semicontinuous convex function. We can now state the main result of the paper.

Theorem 10 *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $T : X \rightrightarrows X^*$ be a set-valued operator satisfying*

$$\operatorname{Gr} T \subset \operatorname{Gr} \partial g. \tag{7}$$

(In particular T is cyclically monotone.) Let $x_0 \in \operatorname{dom} T$. Denote by h the proper lower semicontinuous convex function defined by relation (6). Then the following assertions hold.

(A1) If $\operatorname{int}(\operatorname{dom} g) \neq \emptyset$ and $\operatorname{dom} T$ is dense in $\operatorname{int}(\operatorname{dom} g)$, then

$$g - g(x_0) = h \tag{8}$$

on $\operatorname{dom} g$.

(A2) If $\operatorname{int}(\operatorname{dom} g^*) \neq \emptyset$ and $\operatorname{Im} T$ is dense in $\operatorname{int}(\operatorname{dom} g^*)$, then

$$g - g(x_0) = h.$$

Proof Combining (1), (6) and (7) we easily obtain that

$$g - g(x_0) \geq h. \tag{9}$$

(A1) Set $U = \operatorname{int}(\operatorname{dom} g) \neq \emptyset$. In view of (9), we have $U \subset \operatorname{dom} h$. Since U is open, it follows from [9, Proposition 2.5] that

$$U \subset \operatorname{int}(\operatorname{dom} \partial g) \cap \operatorname{int}(\operatorname{dom} \partial h).$$

Hence, by Proposition 8, the maximal monotone operators ∂g and ∂h are minimal w*-cuscos on U. By (7) we have

$$\operatorname{Gr} T \subset \operatorname{Gr} \partial g,$$

while by Proposition 5 we have

$$\mathrm{Gr}\, T \subset \mathrm{Gr}\, \partial h.$$

Since $\mathrm{dom}\, T$ is dense in U, Proposition 9 yields that $\partial g = \partial h$ on U. Consequently (see [10]), there exists $r \in \mathbb{R}$ such that $g = h + r$ on U. A standard argument shows that this last equality can be extended on $\mathrm{dom}\, g$. By definition of h and recalling that the operator T is cyclically monotone we have $h(x_0) = 0$, hence we conclude that $g(x_0) = r$ and thus equality (8) holds as asserted.

(A2) Set $V = \mathrm{int}(\mathrm{dom}\, g^*) \neq \emptyset$. By [9, Theorem 2.28], the operator ∂g^* is locally bounded on V. By (2) we have the inclusion $\mathrm{Gr}\, \left(i \circ (\partial g)^{-1}\right) \subset \mathrm{Gr}\, (\partial g^*)$. Combining with (7) we obtain

$$\mathrm{Gr}\,(i \circ T^{-1}) \subset \mathrm{Gr}\, \partial g^*, \tag{10}$$

which yields that T^{-1} is locally bounded on V. Applying Lemma 6 we obtain

$$V \subset \mathrm{int}(\mathrm{dom}\, h^*). \tag{11}$$

Set now $S = i \circ T^{-1}$. According to relation (10) we have

$$\mathrm{Gr}\, S \subset \mathrm{Gr}\, \partial g^*$$

Furthermore, by Proposition 5 we have $\mathrm{Gr}\, T \subset \mathrm{Gr}\, \partial h$, which implies as before that

$$\mathrm{Gr}\, S \subset \mathrm{Gr}\, \partial h^*.$$

Since $\mathrm{dom}\, S = \mathrm{Im}\, T$ is dense in V, and since both ∂g^* and ∂h^* are minimal w*-cuscos on V, it follows by Proposition 9 that $\partial g^* = \partial h^*$ on V. By [10], there exists $r \in \mathbb{R}$ such that

$$g^* = h^* + r$$

on $\mathrm{int}(\mathrm{dom}\, g^*)$. Since the latter is nonempty, the above equality can be extended to X^*, provided that

$$\mathrm{int}(\mathrm{dom}\, g^*) = \mathrm{int}(\mathrm{dom}\, h^*). \tag{12}$$

Let us now prove this last equality. Taking conjugates in both sides of the inequality in (9) we have $g^* + g(x_0) \leq h^*$, hence, in particular, $\mathrm{dom}\, h^* \subset \mathrm{dom}\, g^*$ and so $\mathrm{int}(\mathrm{dom}\, h^*) \subset \mathrm{int}(\mathrm{dom}\, g^*)$. In view of (11) we conclude that equality (12) holds as desired. It follows that

$$g^* = h^* + r.$$

Taking conjugates and considering the restriction on X we obtain $g = h - r$. Since $h(x_0) = 0$ we conclude that $g(x_0) = -r$ and thus $g - g(x_0) = h$ as asserted. □

Remark 11 Note that equality (8) may not hold for all $x \in X$. Indeed, let $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of the closed segment $[-1, 1]$. If we define the operator T by

$$T(x) = \begin{cases} \{0\}, & \text{if } x \in (-1,1) \\ \emptyset, & \text{if } x \notin (-1,1). \end{cases}$$

and if we take $x_0 = 0$, then $h = 0$. In this case g and h do not differ to a constant on $\mathbb{R}$.

3.1 Application: Representation of convex epi-pointed functions

The following definition will be useful in the sequel.

Definition 12 *A point $x \in \mathrm{dom}\, g$ is called strongly exposed for the proper lower semicontinuous convex function g if*

$$(x, g(x)) \in \exp(\mathrm{epi}\, g).$$

We denote by $\exp g$ the set of strongly exposed points of g.

For every $x \in \exp g$ we denote by $\mathrm{Exp}(g, x)$ the set of all $x^* \in X^*$ satisfying

$$(x^*, -1) \in \mathrm{Exp}(\mathrm{epi}\, g, (x, g(x)),$$

According to relations (4) and (5) we have

$$\mathrm{Exp}(g, x) \subset \partial g(x). \tag{13}$$

We also set

$$\mathrm{Exp}\, g = \bigcup_{x \in \exp g} \mathrm{Exp}(g, x).$$

It may happen that the set of strongly exposed points be empty, for instance when g is a constant function. We shall avoid this situation since, as we shall show $\exp g$ is non-empty in spaces with the Radon-Nikodym property, under the following coercivity assumption that we recall from [3, p. 1669].

Definition 13 *A proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called epi-pointed if*

$$\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset.$$

The above definition is in fact equivalent to the following coercivity condition:

$$\text{there exist } x^* \in X^*, \rho > 0 \text{ and } r \in \mathbb{R} \text{ such that } f(x) \geq \langle x^*, x\rangle + \rho\|x\| + r \ \text{ for all } x \in X.$$

This has been established in [3, Proposition 4.5] in finite dimensions. Only minor modifications are needed for the general case.

Remark 14 A proper lower semicontinuous function f is epi-pointed if, and only if, $\overline{\mathrm{co}} f$ is epi-pointed.

Let us now state the following consequence of Proposition 3.

Proposition 15 *The set* $\mathrm{Exp}\, g$ *is dense in* $\mathrm{int}(\mathrm{dom}\, g^*)$ *if the Banach space X has the Radon-Nikodym property and the convex function g is epi-pointed.*

Proof Let $x^* \in \mathrm{int}(\mathrm{dom}\, g^*)$ and $\varepsilon > 0$ such that $B(x^*, \varepsilon) \subset \mathrm{int}(\mathrm{dom}\, g^*)$. Set

$$r := \min\left\{1/2,\, \varepsilon(2\|x^*\| + 2)^{-1}\right\}.$$

By relation (3) we have $(x^*, -1) \in \mathrm{int}(\mathrm{dom}\, \sigma_{\mathrm{epi}\, g})$. By Proposition 3, there exists $z^* \in B(x^*, r)$ and $s \in (1-r, 1+r)$, such that $(z^*, -s) \in \mathrm{Exp}(\mathrm{epi}\, g)$. Then obviously $(s^{-1}z^*, -1) \in \mathrm{Exp}(\mathrm{epi}\, g)$, that is $s^{-1}z^* \in \mathrm{Exp}\, g$. A direct calculation now yields

$$\|s^{-1}z^* - x^*\| \leq \|s^{-1}z^* - z^*\| + \|z^* - x^*\| < s^{-1} \mid 1 - s \mid \|z^*\| + r \leq 2r(\|x^*\| + r) + r \leq \varepsilon,$$

that is $s^{-1}z^* \in \mathrm{Exp}\, g \cap B(x^*, \varepsilon)$. This completes the proof. □

We are ready to state the following subdifferential representation result for epi-pointed functions.

Theorem 16 *Suppose that Banach space X has the Radon-Nikodym property and the convex function g is epi-pointed. Let $x_0 \in \mathrm{dom}\, \partial g$. Then for every $x \in X$ we have*

$$g(x) - g(x_0) = \sup\left\{\sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle \ + \ \langle x_n^*, x - x_n\rangle\right\}, \tag{14}$$

where the supremum is taken over all integers n, all $x_1, ..., x_n$ in $\exp g$, and all $x_0^* \in \partial g(x_0)$, $x_1^* \in \partial g(x_1), \ldots, x_n^* \in \partial g(x_n)$.

Proof Let us consider the set-valued operator $T : X \rightrightarrows X^*$ defined for all $x \in X$ by

$$T(x) = \begin{cases} \partial g(x), & \text{if} \quad x \in \{x_0\} \cup \exp g \\ \emptyset, & \text{if} \quad x \notin \{x_0\} \cup \exp g. \end{cases}$$

Since $\operatorname{Gr} T \subset \operatorname{Gr} \partial g$, the operator T is also cyclically monotone.

We claim that the right part of (14) coincides up to a constant with the Rockafellar integration formula (6) for the operator T. Indeed, given an integer $n \geq 1$ and a finite sequence $x_1, ..., x_n$ in $\operatorname{dom} T$ denote by i_0 the smaller index in $\{0, \ldots, n\}$ such that $x_i \neq x_0$ for all $i > i_0$. Then $x_{i_0} = x_0$. Using the cyclic monotonicity of T we have

$$\sum_{i=0}^{i_0} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$

Omitting the terms that do not contribute to the supremum, the sequence $x_1, ..., x_n$ in $\operatorname{dom} T$ can be replaced by the sequence $x_{i_0+1}, ..., x_n$ in $\exp g$.

According to relation (5), we have

$$\operatorname{Exp} g \subset \bigcup_{x \in \exp g} \partial g(x) \subset \operatorname{Im} T.$$

Hence by Proposition 15, Im T is dense in $\operatorname{int}(\operatorname{dom} g^*)$, and the result follows from Theorem 10. □

Remark 17 Formula (14) fails for non-epi-pointed functions, even in finite dimensions. Consider for instance the proper lower semicontinuous convex function $g : \mathbb{R}^2 \to \mathbb{R}$ defined for $(x, y) \in \mathbb{R}^2$ by

$$g(x, y) = \frac{1}{2} y^2.$$

In this case $\exp g = \emptyset$ and for $x_0 = (0, 0)$ formula (14) yields $g(x) = 0$, which is not true.

Remark 18 Formula (14) also fails in Banach spaces without the Radon-Nikodym property. Indeed let $X = c_0(\mathbb{N})$ and let g be the indicator function of the closed unit ball of X. Then g is a proper lower semicontinuous convex function which is also epi-pointed, since g^* coincides with the norm of $X^* = \ell^1(\mathbb{N})$. Let further $x_0 = 0$ and note that $\partial g(x_0) = \{0\}$. Since the closed unit ball of X has no extreme points, it follows easily that $\exp g = \emptyset$. Thus formula (14) yields $g(x) = 0$, which is again not true.

3.2 Application: convexification of epi-pointed functions

Throughout this section we denote by $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous epi-pointed function and we set

$$g = \overline{\text{co}} f.$$

We easily check that

$$x \in \operatorname{dom} \partial f \implies (g(x) = f(x) \text{ and } \partial g(x) = \partial f(x)). \tag{15}$$

The following lemma gives an interesting particular case where the above situation occurs.

Lemma 19 *Let $x \in \exp g$. Then $g(x) = f(x)$ and $\partial g(x) = \partial f(x)$.*

Proof We set $C := \operatorname{epi} g$, $A := \operatorname{epi} f$ and $u := (x, g(x))$. Note that $g(x) = f(x)$ if, and only if, $u \in A$. Let us suppose, towards a contradiction, that $g(x) < f(x)$, that is $u \notin A$. Since A is closed, there exists $\rho > 0$ such that

$$A \cap B(u, \rho) = \emptyset. \tag{16}$$

By assumption $u \in \exp C$, so there exists $p \in X^* \times \mathbb{R}$ and $\varepsilon > 0$ such that

$$C \cap H \subset B(u, \rho),$$

where H is the open half-space $\{v \in X \times \mathbb{R} : \langle p, v\rangle > \langle p, u\rangle - \varepsilon\}$. Then, recalling that $A \subset C$, relation (16) implies $A \cap H = \emptyset$, or equivalently, taking the closed convex hull of the set A, that $C \cap H = \emptyset$. We obtain a contradiction since $u \in C \cap H$. Consequently, $g(x) = f(x)$. The equality of subdifferentials is now straightforward. $\square$

As a consequence of the above lemma we obtain a representation formula for the closed convex envelope g of an epi-pointed function f based on the Fenchel subdifferential of f.

Corollary 20 *Suppose that the Banach space X has the Radon-Nikodym property. Let $x_0 \in \operatorname{dom} \partial f$. Then for every $x \in X$, we have*

$$\overline{\text{co}} f(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \ + \ \langle x_n^*, x - x_n \rangle \right\}, \tag{17}$$

where the supremum is taken over all integers n, all $x_1, x_2, ..., x_n$ in $\operatorname{dom} \partial f$ and all $x_0^* \in \partial f(x_0), x_1^* \in \partial f(x_1), \ldots, x_n^* \in \partial f(x_n)$.

Proof According to formula (1) and using relations (15), the right hand side of (17) defines a proper lower semicontinuous convex function $\hat{f}$ satisfying $\hat{f} \leq g$ (note that $g(x_0) = f(x_0)$). On the other hand, according to Theorem 16 and Lemma 19, we obtain $\hat{f} \geq g$. This finishes the proof. $\square$

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