A convexity criterion via the De Giorgi slope

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Abstract. Let X be a Banach space and $f \in \mathcal{C}^1(X)$ be bounded from below. We show that if for some $m \geq 1$, the function $x \mapsto \|\nabla f(x)\|^m$ is convex, then f is convex. We also establish a more general version of this result: if f is continuous and bounded from below, then it is convex, provided $x \mapsto s_f(x)^m$ is convex for some $m \geq 1$, where s_f denotes the (De Giorgi) metric slope of f.

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1 Introduction

In [2, Corollary 3.17] the following convexity criterion was established for $\mathcal{C}^2$-smooth functions in a Hilbert space $\mathcal{H}$:

- Let $f \in \mathcal{C}^2(\mathcal{H})$ be bounded below. If $V_2(x) = \|\nabla f(x)\|^2$ is convex, then f is convex.

The aim of this work is to extend this criterion to nonsmooth functions defined on a Banach space X. Notice that in principle, this is a challenging task, since the proof of the aforementioned criterion given in [2], depends heavily on the Hilbertian structure as well as on the $\mathcal{C}^2$-smoothness assumption, since it is based on the study of the second order system $\ddot{x}(t) = \frac{1}{2}\nabla V_2(x)$.

As was the case in recent determination results (see [5, 6, 15, 17]) as well as in recent results relating to Monge solutions of the Hamilton-Jacobi equations ([12, 13]), the modulus of the gradient $\|\nabla f\|$ is now replaced by the local (or metric) slope s_f introduced by De Giorgi (see [1, 7] *e.g.*) and defined as follows:

$$s_f(x) := \limsup_{y \to x} \frac{\max\{f(x) - f(y), 0\}}{\|x - y\|}, \quad \text{for every } x \in X.$$

Our main result (see forthcoming Theorem 2.12) reads as follows:

- Let $f \in \mathcal{C}(X)$ be bounded from below. If $V_m(x) := s_f(x)^m$ is convex, for some $m \geq 1$, then f is convex.

The proof of the above result will be given in the next section. As expected, the approach differs significantly from the one in [2] and borrows from techniques introduced in [8, 12].
In this work, we denote by

$$\mathrm{Crit}(f) = \{s_f = 0\} := \{x \in X : \, s_f(x) = 0\}$$

the set of critical points of f. Notice that this is the set $\{\nabla f = 0\}$ in the smooth case.
A function f is called coercive, if $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$, or equivalently, if the sublevel sets $\{f \leq r\}$, $r \in \mathbb{R}$, are compact.

2 Proof of the main result

For the convenience of the reader, we first present an easy proof in the particular case of a $\mathcal{C}^1$–smooth coercive function in finite dimensions.

2.1 A simple proof in the $\mathcal{C}^1$-setting

Given a $\mathcal{C}^1$-smooth function $f : \mathbb{R}^d \to \mathbb{R}$ and $w \in \mathbb{R}^d$, we denote by γ_w the gradient flow of f starting from w, that is,

$$\begin{cases} \dot{\gamma}_w(t) = -\nabla f(\gamma_w(t)), \quad t \geq 0, \\ \gamma_w(0) = w. \end{cases}$$

If the curve $\{\gamma_w(t)\}_{t\geq 0}$ lies in a compact set (in particular, if f is coercive), then $\operatorname{dist}(\gamma_w(t), \operatorname{Crit} f) \underset{t\to\infty}{\longrightarrow} 0$. In what follows, we shall use the following result.

Lemma 2.1. *Let $f \in \mathcal{C}^1(\mathbb{R}^d)$ be coercive and γ_w the gradient orbit of f starting from $w \in \mathbb{R}^d$. Then, for every sequence $\{t_n\}_{n\geq 1}$ with $t_n \underset{n\to\infty}{\longrightarrow} \infty$, there exist a subsequence $\{t_{n_k}\}_{k\geq 1}$ and $p_w \in \operatorname{Crit}(f)$ such that $\gamma_w(t_{n_k}) \underset{k\to\infty}{\longrightarrow} p_w$.*

Proof. Along the gradient orbit γ_w, we have

$$\frac{d}{dt} f(\gamma_w(t)) = \nabla f(\gamma_w(t)) \cdot \dot{\gamma}_w(t) = -\|\nabla f(\gamma_w(t))\|^2 \leq 0.$$

Thus $f(\gamma_w(t)) \leq f(w)$ for all $t \geq 0$ and the curve γ_w lies in the sublevel set $\{f \leq f(w)\}$, which is compact, since f is coercive. Moreover,

$$\int_0^\infty \|\nabla f(\gamma_w(t))\|^2 \, dt \;\leq\; f(w) - \inf f \;<\; +\infty \, .$$

Since $t \mapsto \|\nabla f(\gamma_w(t))\|^2$ is integrable on $[0, +\infty)$, we infer that $\lim_{t\to\infty} \|\nabla f(\gamma_w(t))\| = 0$. Fix $\{t_n\}_{n\geq 1}$ with $\lim_{n\to\infty} t_n = +\infty$. Since the set $\{f \leq f(w)\}$ is compact, there exist a subsequence $\{t_{n_k}\}$ and a vector $p_w \in \mathbb{R}^d$ such that $\lim_{k\to\infty} \gamma_w(t_{n_k}) = p_w$. Since ∇f is continuous, we obtain $\nabla f(p_w) = 0$ and so $p_w \in \operatorname{Crit}(f)$, which completes the proof. $\square$

We shall also need the following lemma.

Lemma 2.2. *Let $f \in \mathcal{C}^1(\mathbb{R}^d)$ be coercive and assume that $V_2(x) := \|\nabla f(x)\|^2$ is convex. Then $\operatorname{Crit} f$ is nonempty and convex and f is constant there.*

Proof. Coercivity of f yields the nonemptiness of $\operatorname{Crit}(f)$. Notice that the (non-negative) function V_2 is convex and $\operatorname{Crit}(f) = \{V_2 = 0\}$. Therefore $\operatorname{Crit}(f)$ is convex. Take now any $x, y \in \operatorname{Crit}(f)$. Then for every $s \in (0, 1)$, we have $x + s(y - x) \in \operatorname{Crit}(f)$ and

$$\frac{d}{ds} f(x + s(y - x)) = \underbrace{\nabla f(x + s(y - x))}_{=0} \cdot (y - x) = 0.$$

It follows that $f(x) = f(y)$ yielding that f is constant on $\operatorname{Crit}(f)$. $\square$

We are now ready to establish our main result in the particular case where f is $\mathcal{C}^1$-smooth and coercive.

Proposition 2.3. *Let $f \in \mathcal{C}^1(\mathbb{R}^d)$ be coercive. Assume that $V_2(x) := \|\nabla f(x)\|^2$ is convex. Then, f is convex.*

Proof. Fix $x, y \in \mathbb{R}^d$. Denote by $\gamma_x, \gamma_y : [0, +\infty) \to \mathbb{R}^d$ the gradient flows starting at x and y, respectively, that is,

$$\begin{cases} \dot{\gamma_x}(t) = -\nabla f(\gamma_x(t)), \ t > 0 \\ \gamma_x(0) = x \end{cases} \quad \text{and} \quad \begin{cases} \dot{\gamma_y}(t) = -\nabla f(\gamma_y(t)), \ t > 0 \\ \gamma_y(0) = y \end{cases}.$$

Set

$$\begin{aligned} \mu(t) &:= \frac{\gamma_x(t) + \gamma_y(t)}{2}, \\ D(t) &:= \frac{1}{2}\Big(f(\gamma_x(t)) + f(\gamma_y(t))\Big) - f(\mu(t)). \end{aligned}$$

We first prove that the function $t \mapsto D(t)$ is nonincreasing. First observe that

$$\frac{d}{dt} f(\gamma_z(t)) = \langle \nabla f(\gamma_z(t)), \dot{\gamma}_z(t) \rangle = -\|\nabla f(\gamma_z(t))\|^2 = -V_2(\gamma_z(t)) \quad \text{for } z \in \{x, y\} \text{ and } t > 0. \tag{2.1}$$

It follows that

$$D'(t) = -\frac{1}{2}\Big(V_2(\gamma_x(t)) + V_2(\gamma_y(t))\Big) + \Big\langle \nabla f(\mu(t)), \frac{\nabla f(\gamma_x(t)) + \nabla f(\gamma_y(t))}{2} \Big\rangle. \tag{2.2}$$

On the one hand, using the assumption that V_2 is convex, we obtain

$$\|\nabla f(\mu(t))\| = \sqrt{V_2(\mu(t))} \leq \sqrt{\frac{V_2(\gamma_x(t)) + V_2(\gamma_y(t))}{2}}. \tag{2.3}$$

On the other hand, using the convexity of the map $w \mapsto \|w\|^2$, we have

$$\left\| \frac{\nabla f(\gamma_x(t)) + \nabla f(\gamma_y(t))}{2} \right\| \leq \sqrt{\frac{V_2(\gamma_x(t)) + V_2(\gamma_y(t))}{2}}. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we conclude that $D' \leq 0$ and hence it is nonincreasing. It follows that

$$D(0) \geq D(t) \quad \text{for every } t \geq 0. \tag{2.5}$$

We next prove that there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} D(t_n) = 0$. Applying Lemma 2.1 for $w = x$ and subsequently, for $w = y$, we obtain a sequence $\{t_n\}_{n\geq 1}$ and $p_x, p_y \in \mathrm{Crit}(f) = \{\nabla f = 0\}$ such that

$$\gamma_x(t_n) \underset{n\to\infty}{\longrightarrow} p_x \quad \text{and} \quad \gamma_y(t_n) \underset{n\to\infty}{\longrightarrow} p_y.$$

Consequently, thanks to the continuity of f and the fact that f is constant on the convex set $\mathrm{Crit}(f)$ (see Lemma 2.2), we deduce

$$\lim_{n\to\infty} D(t_n) = \frac{1}{2}(f(p_x) + f(p_y)) - f\left(\frac{p_x + p_y}{2}\right) = 0.$$

To conclude, combining this with (2.5), we get $D(0) \geq \lim_{n\to\infty} D(t_n) = 0$. Therefore,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)).$$

Since f is continuous, we infer that f is convex, which completes the proof. $\square$

2.2 General case: detecting convexity from the slope in Banach spaces

In this subsection we shall prove the general case: f is merely assumed continuous and bounded from below and we work in an arbitrary Banach space.

We shall first need the following result.

Lemma 2.4. *Let X be a Banach space and let $f \in \mathcal{C}(X)$ be such that $s_f \in \mathcal{C}(X)$. Let $\xi : [a,b] \to X$ be an absolutely continuous curve. Then, it holds*

$$f(\xi(s)) - f(\xi(t)) \le \int_s^t s_f(\xi(\tau)) \|\dot{\xi}(\tau)\| d\tau, \quad \textit{for every } a \le s \le t \le b. \tag{2.6}$$

Proof. First, consider the case in which $\xi : [a,b] \to X$ a 1–Lipschitz curve. In this case, the inequality (2.6) can be derived by combining [12, Proposition 4.5] and [8, Proposition 2.6], via the theory of eikonal equations in metric spaces. In the sequel, however, we give a self-contained and direct proof based on the lower Dini derivative, which avoids the machinery of viscosity solutions in general metric spaces.

Consider the function

$$F(t) := f(\xi(t)) + \int_a^t s_f(\xi(\tau))\, d\tau, \quad \text{for } t \in [a,b].$$

Recall that its *lower right Dini derivative* at $t \in (a,b)$ is defined by:

$$D_+F(t) := \liminf_{h \to 0_+} \frac{F(t+h) - F(t)}{h}.$$

Fix $t \in (a,b)$ and $\varepsilon > 0$. By the definition of slope, there exists $\delta > 0$ such that

$$f(\xi(t)) - f(y) \le (s_f(\xi(t)) + \varepsilon)\|\xi(t) - y\| \quad \text{ for every } y \in B_\delta(\xi(t)).$$

Using the fact that ξ is 1–Lipschitz, we have, for every $0 < h < \min\{\delta, b-t\}$,

$$f(\xi(t+h)) - f(\xi(t)) \ge -(s_f(\xi(t)) + \varepsilon)\|\xi(t) - \xi(t+h)\| \ge -(s_f(\xi(t)) + \varepsilon)h.$$

Therefore, we obtain

$$\frac{F(t+h) - F(t)}{h} \ge -(s_f(\xi(t)) + \varepsilon) + \frac{1}{h}\int_t^{t+h} s_f(\xi(\tau))\, d\tau \quad \text{ for every } 0 < h < \min\{\delta, b-t\}.$$

The continuity of $s_f \circ \xi$ then yields

$$D_+F(t) \ge -(s_f(\xi(t)) + \varepsilon) + s_f(\xi(t)) = -\varepsilon.$$

Since $\varepsilon > 0$ and $t \in (a,b)$ are chosen arbitrarily, we infer that $D_+F(t) \ge 0$ for every $t \in (a,b)$. Therefore F is nondescreasing in $[a,b]$ (see e.g. [3, Chapter 11, Corollary 4.2] or [10]). It follows that, for any $a \le s \le t \le b$

$$f(\xi(s)) - f(\xi(t)) \le \int_a^t s_f(\xi(\tau))\, d\tau - \int_a^s s_f(\xi(\tau))\, d\tau = \int_s^t s_f(\xi(\tau))\, d\tau.$$

It remains to prove (2.6) for an arbitrary absolutely continuous curve $\xi : [a,b] \to X$. Define the arc-length function

$$\ell(r) := \int_a^r \|\dot{\xi}(\tau)\| \, d\tau, \qquad \text{for } r \in [a,b],$$

and set $L := \ell(b)$. The case $L = 0$ is vacuous. Assume that $L > 0$. Since ξ is absolutely continuous, ℓ is continuous, nondecreasing and satisfies $\ell'(r) = \|\dot{\xi}(r)\|$ for a.e. $r \in [a,b]$. Since $\ell(r_1) = \ell(r_2)$ implies $\xi(r_1) = \xi(r_2)$, the curve $\eta : [0,L] \to X$ defined by

$$\eta(\ell(r)) := \xi(r), \qquad \text{for } r \in [a,b],$$

is well-defined. Furthermore, if $0 \le \rho_1 \le \rho_2 \le L$, we choose $r_1, r_2 \in [a,b]$ such that $\ell(r_i) = \rho_i$ for $i \in \{1,2\}$. Since ℓ is nondecreasing, we may take $r_1 \le r_2$ and so

$$\|\eta(\rho_2) - \eta(\rho_1)\| = \|\xi(r_2) - \xi(r_1)\| \le \int_{r_1}^{r_2} \|\dot{\xi}(\tau)\| \, d\tau = \rho_2 - \rho_1.$$

Thus η is 1-Lipschitz. Applying the result already proved for 1-Lipschitz curves, we obtain, for any $a \le s \le t \le b$,

$$\begin{aligned} f(\xi(s)) - f(\xi(t)) = f(\eta(\ell(s))) - f(\eta(\ell(t))) &\le \int_{\ell(s)}^{\ell(t)} s_f(\eta(\rho)) \, d\rho \\ &= \int_s^t s_f(\eta(\ell(\tau)))\ell'(\tau) \, d\tau = \int_s^t s_f(\xi(\tau)) \|\dot{\xi}(\tau)\| \, d\tau, \end{aligned}$$

where we have used that $\ell'(\tau) = \|\dot{\xi}(\tau)\|$ for a.e $\tau \in [a,b]$. This completes the proof. $\square$

Before we proceed, let us register the following immediate consequence of Lemma 2.4.

Corollary 2.5. *Let X be a Banach space, $f \in \mathcal{C}(X)$ and $s_f \in \mathcal{C}(X)$. Assume that K is a nonempty convex subset of X and s_f is bounded on K by a constant $M > 0$. Then f is M-Lipschitz on K.*

Proof. Given $x, y \in K$, apply (2.6) for the 1-Lipschitz curve $\xi(t) := y + t\,(\frac{x-y}{\|x-y\|})$, $t \in \big[0, \|x-y\|\big]$. $\square$

We now recall a result from the theory of eikonal equations on complete length spaces. Roughly speaking, the result states that, under the Monge condition $s_u = \ell$, one can construct local curves starting from any point along which u decreases almost optimally. We shall use this curve construction in the sequel. The following statement is a consequence of Proposition 4.8, Proposition 3.5, and Remark 3.6 of [12].

Proposition 2.6. *Let (X,d) be a complete length space and $\Omega \subset X$ an open set. Assume that ℓ is locally uniformly continuous on X and $\inf_\Omega \ell > 0$. Assume that u is locally Lipschitz on Ω such that*

$$s_u(x) = \ell(x) \quad \textit{for every } x \in \Omega. \tag{2.7}$$

Then, for every $x \in \Omega$, there exists a sufficiently small $\delta > 0$ such that for every $\eta > 0$, there exists a 1–Lipschitz curve $\xi : [0,+\infty) \to X$ satisfying $\xi(0) = x$ and

$$u(x) \ge \int_0^r \ell(\xi(\rho)) \, d\rho + u(\xi(r)) - \eta(1+r), \qquad \textit{for every } 0 \le r \le R,$$

where $R > 0$ denotes the exit time of ξ from $B_\delta(x)$, that is,

$$R := \inf \big\{ r \ge 0 : \xi(r) \notin B_\delta(x) \big\}.$$

We obtain easily from the above the following result, which will be used in the sequel.

Corollary 2.7. *Let X be a Banach space and $f \in \mathcal{C}(X)$. Assume that s_f is locally uniformly continuous. Then, for every $x \in X$ with $s_f(x) > 0$, there exists $\delta > 0$ such that for every $\varepsilon > 0$ there exists a 1–Lipschitz curve $\sigma : [0, +\infty) \to X$ satisfying $\sigma(0) = x$ and*

$$f(x) - f(\sigma(r)) \geq \int_0^r s_f(\sigma(\rho))\, d\rho - \varepsilon \quad \textit{for every } 0 \leq r \leq R,$$

where R is the exit time of σ from $B_\delta(x)$.

Proof. Since the functions f and s_f are continuous and $s_f(x) > 0$, there exists $\delta' > 0$ such that

$$m_f(x) := \inf_{B_{\delta'}(x)} f > -\infty \qquad \text{and} \qquad 0 < \underbrace{\inf_{B_{\delta'}(x)} s_f}_{=:\, m_x} \leq \underbrace{\sup_{B_{\delta'}(x)} s_f}_{=:\, M_x} < +\infty.$$

Consequently, thanks to Corollary 2.5, f is Lipschitz on $B_{\delta'}(x)$ with the Lipschitz constant M_x. Applying Proposition 2.6 to the case $\Omega = B_{\delta'}(x)$, $u = f$ and $\ell = s_f$, there exists $\delta \in (0, \delta')$ such that, for every $\eta > 0$, there exists a 1–Lipschitz curve $\xi : [0, +\infty) \to X$ with $\xi(0) = x$ satisfying

$$f(x) \geq \int_0^r s_f(\xi(\rho))\, d\rho + f(\xi(r)) - \eta(1 + r), \qquad \text{for every } 0 \leq r \leq R_\xi, \tag{2.8}$$

where $R_\xi > 0$ denotes the exit time of ξ from $B_\delta(x)$. Since f is bounded from below, we get, for every $\eta \leq \min\{1, m_x/2\}$,

$$f(x) - m_f(x) + \eta \geq \int_0^{R_\xi} s_f(\xi(\rho))\, d\rho - \eta R_\xi \geq (m_x - \eta) R_\xi \geq \frac{m_x R_\xi}{2}.$$

Hence

$$R_\xi \leq \frac{2(f(x) - m_f(x) + 1)}{m_x}.$$

Since R_ξ is bounded by a quantity that depends only on f and x, it follows from (2.8) that

$$f(x) \geq \inf_\xi \left\{ \int_0^{R_\xi} s_f(\xi(\rho))\, d\rho + f(\xi(R_\xi)) \right\}, \tag{2.9}$$

where the infimum is taken over all 1–Lipschitz curves $\xi : [0, +\infty) \to X$ with $\xi(0) = x$. Fix $\varepsilon > 0$. Then, by (2.9), there exists a 1–Lipschitz curve $\sigma : [0, +\infty) \to X$ such that $\sigma(0) = x$ and

$$f(x) \geq \int_0^{R_\sigma} s_f(\sigma(\rho))\, d\rho + f(\sigma(R_\sigma)) - \varepsilon. \tag{2.10}$$

For any fixed $r \in (0, R_\sigma)$, applying Lemma 2.4 to the case $\xi = \sigma$, we have

$$f(\sigma(r)) - f(\sigma(R_\sigma)) \leq \int_r^{R_\sigma} s_f(\sigma(\rho))\, d\rho. \tag{2.11}$$

Combining (2.10) and (2.11), we obtain

$$f(x) \geq \int_0^r s_f(\sigma(\rho))\, d\rho + f(\sigma(r)) - \varepsilon, \qquad \text{for every } 0 \leq r \leq R_\sigma,$$

which completes the proof. $\square$

Lemma 2.8. *Let X be a Banach space, $m > 1$ and let $f \in \mathcal{C}(X)$ be bounded from below such that s_f is locally uniformly continuous on X. Then, for any $x \in X$ and $\varepsilon > 0$, there exists $\gamma_x : [0, +\infty) \to X$ such that $\gamma_x(0) = x$,*

$$\|\dot{\gamma}_x(t)\| \le s_f^{m-1}(\gamma_x(t)) \quad \textit{for a.e } t > 0, \tag{2.12}$$

and

$$f(x) - f(\gamma_x(t)) \ge \int_0^t s_f^m(\gamma_x(\tau))\, d\tau - \varepsilon, \quad \textit{for every } t > 0. \tag{2.13}$$

Proof. The case $s_f(x) = 0$ is vacuous. Assume that $s_f(x) > 0$. We split the proof into two steps.

Step 1: *Local existence.* Fix $x \in X$ and $\varepsilon > 0$. We prove that there exist $T_x > 0$ depending only on x and a curve $\gamma_x : [0, T_x) \to X$ such that

$$\|\dot{\gamma}_x(t)\| \le s_f^{m-1}(\gamma_x(t)) \quad \text{a.e } t \in (0, T_x), \tag{2.14}$$

and

$$f(x) - f(\gamma_x(t)) \ge \int_0^t s_f^m(\gamma_x(\tau))\, d\tau - \varepsilon, \quad \text{for every } t \in (0, T_x). \tag{2.15}$$

First, let $\delta > 0$ be defined as in Corollary 2.7. Notice that

$$0 < \underbrace{\inf_{y \in B_\delta(x)} s_f(y)}_{:= m_x} \le \underbrace{\sup_{y \in B_\delta(x)} s_f(y)}_{:= M_x} < +\infty.$$

Therefore, thanks to Corollary 2.7, there exists a curve $\sigma_x : [0, +\infty) \to X$ with $\sigma_x(0) = x$ such that $\|\dot{\sigma}_x(r)\| \le 1$ a.e $r > 0$ and

$$f(x) - f(\sigma_x(r)) \ge \int_0^r s_f(\sigma_x(\rho))\, d\rho - \varepsilon \quad \text{for every } r \in (0, R_x). \tag{2.16}$$

Here, R_x denotes the exit time of σ_x from $B_\delta(x)$:

$$R_x := \inf \big\{ r \ge 0 : \sigma_x(r) \notin B_\delta(x) \big\}.$$

Since σ_x is 1–Lipschitz, $R_x \ge \delta$. Define

$$\varphi(r) := \int_0^r \frac{1}{s_f^{m-1}(\sigma_x(\rho))}\, d\rho, \qquad r \in [0, R_x).$$

Since $\inf_{r \in [0, R_x)} s_f(\sigma_x(r)) \ge m_x > 0$, φ is strictly increasing and $\mathcal{C}^1$ on $(0, R_x)$ with

$$\varphi'(r) = \frac{1}{s_f^{m-1}(\sigma_x(r))} \quad \text{for every } r \in (0, R_x).$$

Let $\theta := \varphi^{-1} : [0, T_x) \to [0, R_x)$ be the inverse of φ. Since $R_x > \delta/2$, we have

$$T_x = \lim_{r \nearrow R_x} \varphi(r) \ge \varphi(\delta/2) \ge \frac{\delta}{2 M_x^{m-1}} > 0 \quad \text{where } M_x := \sup_{y \in B_\delta(x)} s_f(y) < +\infty. \tag{2.17}$$

Set $\gamma_x(t) := \sigma_x(\theta(t))$ for $t \in [0, T_x)$. Then $\gamma_x(0) = x$ and for every $t \in (0, T_x)$, we have

$$\theta'(t) = \frac{1}{\varphi'(\theta(t))} = s_f^{m-1}(\sigma_x(\theta(t))) = s_f^{m-1}(\gamma_x(t)).$$

Therefore

$$\|\dot{\gamma}_x(t)\| \leq \|\dot{\sigma}_x(\theta(t))\|\, \theta'(t) \leq s_f^{m-1}(\gamma_x(t)) \quad \text{for a.e. } t \in (0, T_x).$$

It remains to verify inequality (2.15) for $t \in (0, T_x)$. To this end, let $t \in (0, T_x)$ and apply the estimate (2.16) for σ_x at $\theta(t)$, to obtain

$$\begin{aligned} f(x) - f(\gamma_x(t)) = f(x) - f(\sigma_x(\theta(t))) &\geq \int_0^{\theta(t)} s_f(\sigma_x(\rho))\, d\rho - \varepsilon \\ &= \int_0^t s_f(\sigma_x(\theta(\tau)))\theta'(\tau)\, d\tau - \varepsilon = \int_0^t s_f^m(\gamma_x(\tau))\, d\tau - \varepsilon. \end{aligned}$$

Step 2: *Global existence.* We shall prove the existence of a curve $\gamma_x : [0, +\infty) \to X$ with $\gamma_x(0) = x$ satisfying (2.14)–(2.15) for all $t \in [0, +\infty)$ (that is, we can take $T_x = +\infty$).

Fix a continuous strictly increasing function $\kappa : [0, +\infty] \to [0, \varepsilon]$ such that

$$\kappa(0) = 0, \quad \kappa(T) < \varepsilon \quad \text{for every } T < +\infty \quad \text{and} \quad \kappa(+\infty) = \varepsilon.$$

For instance, we can take $\kappa(t) = \varepsilon(1 - e^{-t})$ or $\kappa(t) = \frac{2\varepsilon}{\pi} \arctan t$. Let $\mathcal{A}_x$ be the set of all pairs (T, γ), where $T \in (0, +\infty]$ and $\gamma : [0, T) \to X$ is absolutely continuous such that $\gamma(0) = x$,

$$\|\dot{\gamma}(t)\| \leq s_f(\gamma(t))^{m-1} \quad \text{for a.e. } t \in (0, T), \tag{2.18}$$

and

$$f(x) - f(\gamma(t)) \geq \int_0^t s_f^m(\gamma(s))\, ds - \kappa(T), \qquad \text{for every } t \in (0, T). \tag{2.19}$$

Due to Step 1, we observe that $\mathcal{A}_x$ is nonempty. Indeed, let T_x be defined as in Step 1. Applying the local existence result in Step 1 at x with any error $\eta < \kappa(T_x)$ yields the existence of an admissible pair $(T_x, \gamma) \in \mathcal{A}_x$.

For any $(T, \gamma), (T', \gamma') \in \mathcal{A}_x$, we write

$$(T, \gamma) \preceq (T', \gamma') \quad \text{if } \; T \leq T' \; \text{ and } \; \gamma'\big|_{[0,T)} = \gamma.$$

Observe that $(\mathcal{A}_x, \preceq)$ is partially ordered.

Claim 2.9. *Every chain in $(\mathcal{A}_x, \preceq)$ admits an upper bound.*

Proof of Claim 2.9. Let $\{(T_i, \gamma_i)\}_{i \in I}$ be a chain in $\mathcal{A}_x$. Set $T_* = \sup_{i \in I} T_i \in (0, +\infty]$. By total ordering, the curves are compatible: if $T_i \leq T_j$, then $\gamma_j = \gamma_i$ on $[0, T_i)$. Thus we may define a curve γ_* on $[0, T_*)$ by setting

$$\gamma_*(t) := \gamma_i(t) \quad \text{whenever } t < T_i.$$

It follows directly that γ_* is absolutely continuous, $\gamma_*(0) = x$ and

$$\|\dot{\gamma}_*(t)\| \leq s_f^{m-1}(\gamma_*(t)) \quad \text{for a.e } t \in (0, T_*).$$

The curve γ_* also satisfies the inequality (2.19) for $t \in (0, T_*)$. Indeed, for any fixed $t < T_*$, there exists $i \in I$ such that $t < T_i$. Therefore, we have

$$f(x) - f(\gamma_*(t)) = f(x) - f(\gamma_i(t)) \geq \int_0^t s_f^m(\gamma_*(\tau))\, d\tau - \kappa(T_i) \geq \int_0^t s_f^m(\gamma_*(s))\, ds - \kappa(T_*),$$

where we have used that $\kappa(T_i) \leq \kappa(T_*)$. We have proved that $(T_*, \gamma_*) \in \mathcal{A}_x$. Therefore, every chain in $(\mathcal{A}_x, \preceq)$ admits an upper bound in $\mathcal{A}_x$. $\Diamond$

Using Claim 2.9, it follows from Zorn's lemma that $(\mathcal{A}_x, \preceq)$ has a maximal element $(T_{\max}, \gamma)$. We prove that $T_{\max} = +\infty$. Arguing by contradiction, assume that $T_{\max} < +\infty$. Notice first that for every $0 \leq s < t < T_{\max}$ we have:

$$\|\gamma(s) - \gamma(t)\| \leq \int_s^t \|\dot{\gamma}(\tau)\|\, d\tau \underbrace{\leq}_{(2.18)} \int_s^t s_f^{m-1}(\gamma(\tau))\, d\tau\,.$$

Applying the Hölder inequality for $p = m$ and $q = \frac{m}{m-1}$ (where $\frac{1}{p} + \frac{1}{q} = 1$) we obtain:

$$\int_s^t s_f^{m-1}(\gamma(\tau))\, d\tau \;\leq\; \left(\int_s^t 1^m d\tau\right)^{\frac{1}{m}} \left(\int_s^t s_f^{m-1}(\gamma(\tau))^{\frac{m}{m-1}} d\tau\right)^{\frac{m-1}{m}} = (t-s)^{\frac{1}{m}} \left(\int_s^t s_f^m(\gamma(\tau)) d\tau\right)^{\frac{m-1}{m}}.$$

Finally, using an estimate similar to (2.19) we obtain:

$$\int_s^t s_f^m(\gamma(\tau)) d\tau \leq f(\gamma(s)) - f(\gamma(t)) + \kappa(T_*) \leq f(x) - \inf\, f + \varepsilon\,.$$

Combining the above, we obtain:

$$\|\gamma(s) - \gamma(t)\| \;\leq\; C\,(t-s)^{\frac{1}{m}} \quad \text{where } C := (f(x) - \inf\, f + \varepsilon)^{\frac{m-1}{m}}\,.$$

Since X is complete, the limit $\bar{x} := \gamma(T_{\max}) = \lim_{t \nearrow T_{\max}} \gamma(t)$ exists.

If $s_f(\bar{x}) = 0$, take any $\alpha > 0$ and define $\widetilde{\gamma} : [0, T_{\max} + \alpha) \to X$ by

$$\widetilde{\gamma}(t) := \begin{cases} \gamma(t), & 0 \leq t \leq T_{\max}, \\ \bar{x}\,, & T_{\max} \leq t < T_{\max} + \alpha. \end{cases}$$

One can directly check that $(T_{\max} + \alpha, \widetilde{\gamma}) \in \mathcal{A}_x$, which contradicts the maximality of $(T_{\max}, \gamma)$.

Consider the case $s_f(\bar{x}) > 0$. Let $T_{\bar{x}} > 0$ be defined as in Step 1 (at $\bar{x}$). Fix $0 < \alpha < T_{\bar{x}}$. Since κ is strictly increasing, we may choose

$$0 < \eta < \kappa(T_{\max} + \alpha) - \kappa(T_{\max}).$$

Thanks to Step 1, there exists a curve $\gamma_{\bar{x}} : [0, T_{\bar{x}}) \to X$ such that $\gamma_{\bar{x}}(0) = \bar{x}$ and it satisfies (2.14)–(2.15) with the error η. Define $\widetilde{\gamma} : [0, T_{\max} + \alpha) \to X$ by

$$\widetilde{\gamma}(t) := \begin{cases} \gamma(t), & 0 \leq t \leq T_{\max}, \\ \gamma_{\bar{x}}(t - T_{\max}), & T_{\max} \leq t < T_{\max} + \alpha. \end{cases}$$

We shall prove that $(T_{\max}+\alpha,\widetilde{\gamma})\in\mathcal{A}_x$. Observe first that $\widetilde{\gamma}$ satisfies the estimate (2.18) for a.e $t\in(0,T_{\max}+\alpha)$. It remains to check that it satisfies (2.19) for every $t\in(0,T_{\max}+\alpha)$. It suffices to consider the case $t=T_{\max}+t'$ for any $t'\in[0,\alpha)$. Using the monotone convergence theorem and the continuity of f, we obtain

$$\begin{aligned}
f(x)-f(\widetilde{\gamma}(T_{\max})) &= f(x)-\lim_{t\nearrow T_{\max}} f(\gamma(t)) \\
&\geq \lim_{t\nearrow T_{\max}} \int_0^t s_f^m(\widetilde{\gamma}(\tau))\,d\tau-\kappa(T_{\max}) \geq \int_0^{T_{\max}} s_f^m(\widetilde{\gamma}(\tau))\,d\tau-\kappa(T_{\max}+\alpha).
\end{aligned}$$

Then, we have, for any $t=T_{\max}+t'\in(T_{\max},T_{\max}+\alpha)$

$$\begin{aligned}
f(x)-f(\widetilde{\gamma}(t)) &= f(x)-f(\gamma_{\bar{x}}(t')) \\
&= f(x)-f(\gamma(T_{\max}))+f(\bar{x})-f(\gamma_{\bar{x}}(t')) \\
&\geq \int_0^{T_{\max}} s_f^m(\gamma(\tau))\,d\tau-\kappa(T_{\max})+\int_0^{t'} s_f^m(\gamma_{\bar{x}}(\tau))\,d\tau-\eta \\
&= \int_0^{t} s_f^m(\widetilde{\gamma}(\tau))\,d\tau-(\eta+\kappa(T_{\max})) \\
&\geq \int_0^{t} s_f^m(\widetilde{\gamma}(\tau))\,d\tau-\kappa(T_{\max}+\alpha),
\end{aligned}$$

where the last inequality follows from the choice of η. Thus, we have $(T_{\max}+\alpha,\widetilde{\gamma})\in\mathcal{A}_x$, again contradicting the maximality of $(T_{\max},\gamma)$.

In conclusion, we have proved that the maximal element $(T_{\max},\gamma)$ of $(\mathcal{A}_x,\preceq)$ satisfies $T_{\max}=+\infty$. Using the fact that $\kappa(+\infty)=\varepsilon$, we conclude that the curve γ satisfies the desired estimates. This proves Lemma 2.8. $\square$

Before proceeding to our main result, we shall also need the following proposition.

Proposition 2.10. *Let X be a Banach space. Assume $f\in\mathcal{C}(X)$ and $s_f:X\to\mathbb{R}$ is convex. Then, s_f is continuous (and consequently, locally Lipschitz).*

Proof. Since f is continuous, writing $s_f(x)$ as

$$s_f(x):=\lim_{n\to\infty}\underbrace{\sup_{0<\|h\|<\frac{1}{n}}\left(\frac{\max\{f(x)-f(x+h),0\}}{\|h\|}\right)}_{:=\phi_n(x)\ \text{(lsc function of } x)},\quad \text{for every } x\in X,$$

we deduce that s_f is a Baire-2 function and consequently Baire-measurable [1]. The result follows directly from [14, Theorem 6]. $\square$

[1]Here, a Baire-1 function is a pointwise limit of continuous functions and a Baire-2 function is a pointwise limit of Baire-1 functions. In metric spaces, lower semicontinuous functions are Baire-1. A real-valued function $g:X\to\mathbb{R}$ is Baire-measurable if, for every nontrivial open interval $I\subset\mathbb{R}$, there is an open set $O\subset X$ such that the symmetric difference $g^{-1}(I)\triangle O$ is a countable union of nowhere dense subsets of X.

Remark 2.11 (continuity of convex functions). It is well-known that in a finite dimensional space, every convex function f with values in $\mathbb{R}$ is continuous. In every infinite dimensional Banach space, there exist convex (even linear) real-valued functions, which are discontinuous. In all these examples, the functions fail to be Baire-measurable.

One can provide a direct proof (communicated to us by G. Godefroy) of the fact that every Baire-measurable convex function f from a Banach space X to $\mathbb{R}$ is continuous. Indeed, let $x \in X$ be an arbitrary point. By a standard argument (change of coordinates), we can assume that $x = 0$ and $f(0) = 0$. Setting $C_n := \{f \leq n\}$, for $n \geq 1$, we have that C_n is convex, absorbing and Baire (by assumption). Since $X = \bigcup_{n \geq 1} C_n$, there exists some $n_0 \in \mathbb{N}$ such that C_{n_0} is of second category. By [9, Lemme VI.4.2] C_{n_0} is a neighborhood of 0 and continuity of f follows.

Let us finally recall that completeness of the space X is essential: indeed, taking $X = (c_{00}(\mathbb{N}), \|\cdot\|_2)$ (the space of eventually null sequences equipped with the 2-norm) we see that the lower semicontinuous convex function $f(x) = \|x\|_1 := \sum_{n \in \mathbb{N}} |x_n|$, for $x = (x_n)_n \in c_{00}(\mathbb{N})$, is nowhere continuous.

We are now ready to establish the main result of this work.

Theorem 2.12. (convexity criterion via slope) *Let X be a Banach space and let $f \in \mathcal{C}(X)$ be bounded from below. Assume that for some $m \geq 1$, the function $V_m := s_f^m : X \to \mathbb{R}$ is convex. Then, f is convex.*

Remark 2.13. The assumptions in Theorem 2.12 are essential and the criterion is not reversible.

(i). The continuity assumption on f cannot, in general, be weakened to lower semicontinuity. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2, & x \neq 0, \\ -1\,, & x = 0. \end{cases}$$

Then f is lower semicontinuous, bounded from below and nonconvex. However $s_f^2(x) = 4x^2$ is smooth and convex.

(ii). The assumption that f is bounded from below is essential and cannot be omitted. For instance, the function $f(x, y) = x^2 - y^2$ is continuous and nonconvex but is not bounded from below. Moreover, $\|\nabla f(x, y)\|^2 = 4(x^2 + y^2)$ is convex. See also [2, Remark 3.18] for other examples.

(iii). The converse implication in Theorem 2.12 is false. Namely, convexity of f does not imply convexity of s_f^m. Consider

$$f(x) = \sqrt{1 + x^2}, \quad x \in \mathbb{R}.$$

Then f is convex, C^1, and bounded from below. However,

$$s_f^m(x) = |f'(x)|^m = \frac{|x|^m}{(1 + x^2)^{m/2}}, \quad x \in \mathbb{R},$$

is not convex for any $m \geq 1$.

Proof of Theorem 2.12. Notice that if the function s_f is convex, then so is the function s_f^m, for every $m \geq 1$. Therefore, it suffices to establish the result for $m > 1$.

Notice that according to Proposition 2.10, the convex function s_f^m is continuous and consequently, locally Lipschitz. Fix $x, y \in X$ and $\varepsilon > 0$. Thanks to Lemma 2.8, for each $z \in \{x, y\}$, there exists a curve $\gamma_z : [0, +\infty) \to X$ such that $\gamma_z(0) = z$,

$$\|\dot{\gamma_z}(t)\| \leq s_f^{m-1}(\gamma_z(t)) \quad \text{for a.e } t > 0, \tag{2.20}$$

and

$$f(z) - f(\gamma_z(t)) \geq \int_0^t V_m(\gamma_z(\tau))\, d\tau - \varepsilon \quad \text{for every } t > 0. \tag{2.21}$$

Set

$$\mu(t) := \frac{\gamma_x(t) + \gamma_y(t)}{2} \qquad \text{and} \qquad D(t) := \frac{1}{2}\Big(f(\gamma_x(t)) + f(\gamma_y(t))\Big) - f(\mu(t)).$$

We first prove that for every $t > 0$, one has $D(t) - 2\varepsilon \leq D(0)$. Indeed, it follows from (2.21) that

$$D(t) - D(0) \leq -\frac{1}{2}\int_0^t \Big(V_m(\gamma_x(\tau)) + V_m(\gamma_y(\tau))\Big)\, d\tau + 2\varepsilon + f(\mu(0)) - f(\mu(t)).$$

Applying Lemma 2.4 to the curve $\mu : [0,t] \to X$, we obtain

$$f(\mu(0)) - f(\mu(t)) \leq \int_0^t s_f(\mu(\tau))\|\dot{\mu}(\tau)\|\, d\tau.$$

Hölder's inequality then yields

$$\int_0^t s_f(\mu(\tau))\|\dot{\mu}(\tau)\|\, d\tau \leq \left(\int_0^t V_m(\mu(\tau))\, d\tau\right)^{\frac{1}{m}} \left(\int_0^t \|\dot{\mu}(\tau)\|^{\frac{m}{m-1}}\, d\tau\right)^{\frac{m-1}{m}}.$$

On the one hand, it follows from the convexity of V_m that

$$\int_0^t V_m(\mu(\tau))\, d\tau \;\leq\; \frac{1}{2}\int_0^t V_m(\gamma_x(\tau)) + V_m(\gamma_y(\tau))\, d\tau.$$

On the other hand, recall that for each $z \in \{x, y\}$, one has $\|\dot{\gamma}_z\| \leq s_f^{m-1}(\gamma_z(t))$ for a.e $t > 0$. Hence the convexity of the map $w \mapsto \|w\|^{\frac{m}{m-1}}$ yields

$$\int_0^t \|\dot{\mu}(\tau)\|^{\frac{m}{m-1}}\, d\tau \leq \frac{1}{2}\int_0^t \Big(\|\dot{\gamma}_x(\tau)\|^{\frac{m}{m-1}} + \|\dot{\gamma}_x(\tau)\|^{\frac{m}{m-1}}\Big)\, d\tau \leq \frac{1}{2}\int_0^t V_m(\gamma_x(\tau)) + V_m(\gamma_y(\tau))\, d\tau.$$

Therefore,

$$f(\mu(0)) - f(\mu(t)) \leq \frac{1}{2}\int_0^t V_m(\gamma_x(\tau)) + V_m(\gamma_y(\tau))\, d\tau$$

which yields

$$D(t) - D(0) \leq 2\varepsilon. \tag{2.22}$$

We now quantify the difference between $f(\mu(t))$ and $f(\gamma_z(t))$ for each $z \in \{x, y\}$. Fix $z \in \{x, y\}$ and $t > 0$. Applying Lemma 2.4 to the straight-line segment connecting $\gamma_z(t)$ and $\mu(t)$, we get

$$\begin{aligned} f(\gamma_z(t)) - f(\mu(t)) &\leq \int_0^1 s_f(\underbrace{\mu(t) + \tau(\gamma_z(t) - \mu(t))}_{\xi(\tau)})\, \|\underbrace{\gamma_z(t) - \mu(t)}_{\dot{\xi}(\tau)}\|\, d\tau \\ &= \int_0^1 V_m(\mu(t) + \tau(\gamma_z(t) - \mu(t))^{1/m}\|\gamma_z(t) - \mu(t)\|\, d\tau \\ &\leq (V_m(\gamma_x(t)) + V_m(\gamma_y(t)))^{1/m}\, \|\gamma_z(t) - \mu(t)\|, \end{aligned} \tag{2.23}$$

where convexity of V_m is used to obtain the last inequality. Furthermore, it follows from (2.20) and Hölder's inequality that

$$\|\gamma_z(t) - z\| \leq \int_0^t \|\dot{\gamma_z}(\tau)\| \, d\tau \leq t^{1/m} \left(\int_0^t s_f^m(\gamma_z(\tau)) \, d\tau \right)^{\frac{m-1}{m}} \leq C_z t^{1/m},$$

where $C_z := \big(f(z) - \inf f + \varepsilon\big)^{(m-1)/m}$. Therefore, there exists a constant $C > 0$, depending on f, x, y and $\varepsilon > 0$, such that

$$\|\gamma_z(t) - \mu(t)\| \leq C t^{1/m} \quad \text{for every } t > 0. \tag{2.24}$$

Combining (2.23) and (2.24), we arrive at

$$f(\gamma_z(t)) - f(\mu(t)) \leq C t^{1/m} \left(V_m(\gamma_x(t)) + V_m(\gamma_y(t)) \right)^{1/m}.$$

By a similar argument, we obtain

$$\big| f(\gamma_z(t)) - f(\mu(t)) \big| \leq C t^{1/m} \left(V_m(\gamma_x(t)) + V_m(\gamma_y(t)) \right)^{1/m} \quad \text{for every } t > 0 \text{ and } z \in \{x, y\}. \tag{2.25}$$

To continue, we need the following fundamental fact.

Claim 2.14. *Let $h \in L^1(\mathbb{R}_+, \mathbb{R}_+)$. Then there exists a sequence $\{t_n\}_n$ such that*

$$t_n \underset{n\to\infty}{\longrightarrow} \infty \quad \textit{and} \quad t_n \, h(t_n) \underset{n\to\infty}{\longrightarrow} 0.$$

Proof of Claim 2.14. Arguing by contradiction, assume that there exists $\varepsilon > 0$ and $T > 0$ such that

$$th(t) > \varepsilon \ \text{ for every } t > T.$$

It follows that

$$\int_T^\infty h(t) \, dt \geq \varepsilon \int_T^\infty \frac{dt}{t} = +\infty,$$

which contradicts the fact that h is integrable. ◊

Thanks to (2.21), we observe that

$$\int_0^\infty V_m(\gamma_x(t)) + V_m(\gamma_y(t)) \, dt < +\infty.$$

Therefore, applying Claim 2.14 to the case $h(t) = V_m(\gamma_x(t)) + V_m(\gamma_y(t))$, there exists $\{t_n\}$ such that $t_n \underset{n\to\infty}{\longrightarrow} \infty$ and

$$t_n \Big(V_m(\gamma_x(t_n)) + V_m(\gamma_y(t_n)) \Big) \to 0 \text{ as } n \to \infty.$$

Substituting $t = t_n$ into (2.25), we obtain

$$|D(t_n)| \leq \frac{1}{2} \Big(|f(\gamma_x(t_n)) - f(\mu(t_n))| + |f(\gamma_y(t_n)) - f(\mu(t_n))| \Big) \to 0 \quad \text{as } n \to \infty.$$

We have proved that there exists a sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} D(t_n) = 0. \tag{2.26}$$

To conclude, combining (2.22) and (2.26), we get $D(0) \geq -2\varepsilon$. Since $\varepsilon > 0$ is chosen arbitrarily, we deduce $D(0) \geq 0$, equivalently

$$f\left(\frac{x+y}{2} \right) \leq \frac{1}{2} (f(x) + f(y)).$$

Therefore, f is convex, which completes the proof. □

2.3 Inducing regularity from the slope

In this last subsection, we complete the previous result by observing that in the Hilbert case, continuity of the slope mapping s_f induces some extra regularity on f. This is the aim of the following proposition.

Proposition 2.15. *Let $\mathcal{H}$ be a Hilbert space and $f : \mathcal{H} \to \mathbb{R}$ be a lower semicontinuous convex function with continuous slope s_f. Then, $f \in \mathcal{C}^1(\mathcal{H})$.*

Proof. We first prove that the convex subdifferential ∂f is a singleton at every point in $\mathcal{H}$. Indeed, arguing by contradiction, we assume that there exists $\bar{x} \in \mathcal{H}$ such that $\partial f(\bar{x})$ is not a singleton. Then, there exists $\xi_0 \in \partial f(\bar{x})$ with $\xi_0 \neq \xi_{\bar{x}}$, where $\{\xi_{\bar{x}}\} = \operatorname{argmin}\big\{\|\xi\| : \xi \in \partial f(\bar{x})\big\}$. Consequently,

$$\|\xi_0\| > \|\xi_{\bar{x}}\| = \min_{\xi \in \partial f(\bar{x})} \|\xi\| = s_f(\bar{x}). \tag{2.27}$$

Denote $e := \xi_0/\|\xi_0\|$. For any fixed $t > 0$, it follows from the monotonicity of ∂f that

$$\langle \eta - \xi_0, te\rangle \geq 0 \ \text{ for every } \eta \in \partial f(\bar{x} + te).$$

Hence, using the Cauchy–Schwarz inequality and the fact that $\|e\| = 1$, we get

$$\|\eta\| \geq \langle \eta, e\rangle \geq \langle \xi_0, e\rangle = \|\xi_0\| \ \text{ for every } \eta \in \partial f(\bar{x} + te).$$

Taking the infimum with respect to η, we obtain

$$s_f(\bar{x} + te) \geq \|\xi_0\| \ \text{ for every } t > 0,$$

which, thanks to the continuity of slope, leads to

$$s_f(\bar{x}) = \lim_{t \to 0^+} s_f(\bar{x} + te) \geq \|\xi_0\|,$$

which contradicts (2.27). Therefore, ∂f is singleton at every point in $\mathcal{H}$.

To conclude, we will prove that $\nabla f(\cdot)$ is continuous. Let $\{x_n\} \subset \mathcal{H}$ and $x \in \mathcal{H}$ be such that $x_n \to x$ as $n \to \infty$. It follows that $\|\nabla f(x_n)\| \to \|\nabla f(x)\|$ as $n \to \infty$. Hence $\{\nabla f(x_n)\}_n$ is bounded and so there exist $\xi \in \mathcal{H}$ such that, up to a subsequence, $\nabla f(x_n) \rightharpoonup \xi$ as $n \to \infty$. One can show that $\xi \in \partial f(x)$ and hence $\xi = \nabla f(x)$. Since $\{\nabla f(x_n)\}$ converges weakly and in norm to $\nabla f(x)$, we infer that it strongly converge to $\nabla f(x)$. Proposition 2.15 is proven. $\square$

As a direct consequence of Theorem 2.12 and Proposition 2.15, we obtain the following result.

Corollary 2.16. *Let $\mathcal{H}$ be a Hilbert space. Let $f \in \mathcal{C}(\mathcal{H})$ be a bounded from below such that s_f^m is convex, for some $m \geq 1$. Then, f is a $\mathcal{C}^1$ convex function on $\mathcal{H}$.*

We shall now show that the above result is essentially optimal, in the sense that we cannot obtain better regularity on f, even if we assume more regularity on s_f.

Proposition 2.17. *Let $\beta \in (0, 1]$. Then, the following assertions hold true:*

(i) If $f \in \mathcal{C}^1(\mathbb{R})$ and its slope $|f'|$ is locally β–Hölder, then f is of class $\mathcal{C}^{1,\beta}_{\mathrm{loc}}(\mathbb{R})$.

(*ii*) *There exists a convex $\mathcal{C}^1$ function $f : \mathbb{R}^2 \to [0, +\infty)$ such that $\|\nabla f\|$ is locally β–Hölder but the gradient ∇f fails to be β–Hölder around the origin.*

Proof. (i) Fix $\bar{x} \in \mathbb{R}$ and choose $\delta > 0$ sufficiently small. Set $I := [\bar{x} - \delta, \bar{x} + \delta]$. Since $|f'|$ is locally β–Hölder, there exists $C_I > 0$ such that

$$\big||f'|(x) - |f'|(y)\big| \le C_I |x - y|^\beta \qquad \text{for every } x, y \in I.$$

We claim that f' is β–Hölder on I. Fix $x, y \in I$. The case $f'(x) \cdot f'(y) = 0$ is immediate. If $f'(x)$ and $f'(y)$ have the same sign, then

$$|f'(x) - f'(y)| = \big||f'|(x) - |f'|(y)\big| \le C_I |x - y|^\beta.$$

It remains to consider the case in which $f'(x)$ and $f'(y)$ have opposite signs. By the continuity of f', there exists z between x and y such that $f'(z) = 0$. Hence

$$|f'(x)| \le C_I |x - z|^\beta \text{ and } |f'(y)| \le C_I |y - z|^\beta$$

The concavity of the map $r \mapsto r^\beta$ implies that

$$|x - z|^\beta + |y - z|^\beta \le 2^{1-\beta}(|x - z| + |y - z|)^\beta = 2^{1-\beta}|x - y|^\beta.$$

Therefore,

$$|f'(x) - f'(y)| \le |f'(x)| + |f'(y)| \le 2^{1-\beta} C_I |x - y|^\beta.$$

Thus $f' \in \mathcal{C}^{0,\beta}(I)$. Since $\bar{x}$ was arbitrary, we conclude that $f \in \mathcal{C}^{1,\beta}_{\text{loc}}(\mathbb{R})$.

(ii) Fix $0 < \beta \le 1$ and define

$$f(x, y) = e^x + \frac{1}{1 + \beta/2}|y|^{1+\beta/2}, \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

Observe first that $f \ge 0$, $f \in \mathcal{C}^1(\mathbb{R}^2)$ and is convex. Also a direct computation yields

$$\nabla f(x, y) = \big(e^x, \operatorname{sgn}(y)|y|^{\beta/2}\big) \quad \text{and} \quad V(x, y) := \|\nabla f(x, y)\| = \big(e^{2x} + |y|^\beta\big)^{1/2}.$$

It is straightforward to see that ∇f is locally $(\beta/2)$–Hölder and fails to be β–Hölder around the origin.

We claim that V is locally β–Hölder. Fix $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ and let $K \subset \mathbb{R}^2$ be a compact neighborhood of $(\bar{x}, \bar{y})$. For $z = (x, y)$ and $z' = (x', y') \in K$, we have

$$|V(z) - V(z')| = \Big|\big(e^{2x} + |y|^\beta\big)^{1/2} - \big(e^{2x'} + |y'|^\beta\big)^{1/2}\Big| = \frac{\big|e^{2x} - e^{2x'} + |y|^\beta - |y'|^\beta\big|}{\big(e^{2x} + |y|^\beta\big)^{1/2} + \big(e^{2x'} + |y'|^\beta\big)^{1/2}}.$$

Since $e^{2x} \ge 1$ for all $x \in \mathbb{R}$, we get

$$|V(z) - V(z')| \le \frac{1}{2}|e^{2x} - e^{2x'}| + \frac{1}{2}\big||y|^\beta - |y'|^\beta\big|.$$

Note that $x \mapsto e^{2x}$ is smooth in $\mathbb{R}$ and for $0 < \beta \le 1$, the map $t \mapsto |t|^\beta$ is globally β–Hölder on $\mathbb{R}$. Therefore, there exists $C_K > 0$ such that

$$|V(z) - V(z')| \le C_K\big(|x - x'|^\beta + |y - y'|^\beta\big) \le C_K |z - z'|^\beta \qquad \text{for all } z, z' \in K.$$

Hence $V = \|\nabla f\|$ is locally β–Hölder. This completes the proof. $\square$

Remark 2.18.

(i) For $\beta \in (1/2, 1]$, the function in Proposition 2.17–(ii) can even be chosen to be bounded from below and to have convex squared slope. Indeed, fix $p \in [1, 2\beta)$ and set

$$q(y) := \operatorname{sgn}(y)|y|^{p/2}\sqrt{1+|y|^p} \quad \text{and} \quad f(x,y) := e^x + \int_0^y q(t)\,dt.$$

Then $f \geq 0$, $f \in \mathcal{C}^1(\mathbb{R}^2)$ and f is convex, since e^x is convex and q is nondecreasing. A direct computation yields

$$\nabla f(x,y) = (e^x, q(y)) \quad \text{and} \quad \|\nabla f(x,y)\|^2 = e^{2x} + |y|^p + |y|^{2p}.$$

Hence $\|\nabla f\|^2$ is convex and the slope $\|\nabla f\|$ is locally Lipschitz, whose proof is similar to that of Proposition 2.17–(ii).

However, since $p/2 < \beta$, we have

$$\frac{|q(y) - q(0)|}{|y|^\beta} = |y|^{p/2-\beta}\sqrt{1+|y|^p} \longrightarrow +\infty \qquad \text{as } y \to 0.$$

Therefore ∇f is not β–Hölder at the origin.

(ii) Proposition 2.15 shows that Hölder regularity of $\|\nabla f\|$ cannot, in general, be transferred to Hölder regularity of ∇f. This is in sharp contrast to the rigidity phenomena for the classical eikonal equation

$$\|\nabla f\| \equiv 1,$$

as studied by Caffarelli–Crandall [4] and Ignat [11]. The point is that these results use the constant slope to establish the $\mathcal{C}^{1,1}_{\mathrm{loc}}$ estimates for solutions, not merely regularity of the scalar field $\|\nabla f\|$. Assertion (ii) shows that once the identity $\|\nabla f\| \equiv 1$ is replaced by the weaker assumption $\|\nabla f\| \in \mathcal{C}^{0,\beta}_{\mathrm{loc}}$, the regularity of the slope no longer controls the regularity of the gradient.

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