

Generalized Abelian Theorems for Long Run Optimal Control

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Abstract

We study optimal control problems involving an average defined as the mean of a Lagrangian with respect to a probability measure on $\mathbb{R}_+$. A particular example is the Cesaro average over a fixed time horizon. A fundamental and widely studied question is whether the corresponding value functions converge uniformly as the horizon tends to infinity. Classically, an Abelian theorem states that the uniform convergence of Cesaro means implies the convergence of Abel means to the same limit. We extend this result to value functions associated with more general families of probability measures on $\mathbb{R}_+$. This framework captures long-term optimization problems and the asymptotic behavior of value functions as the time horizon tends to infinity. We introduce suitable conditions on sequences of probability measures that ensure the convergence of the associated value functions in the topology of uniform convergence. Finally, we establish an Abelian-type theorem showing that, whenever the Cesaro values converge uniformly, the values associated with various other sequences of probability measures converge to the same limit.

Key words. Long-Run Optimal Control, Value Function, generalized Abelian theorem.

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1 Introduction

We consider the following control system on $\mathbb{R}^d$

$$(1) \quad \begin{cases} \dot{x}(t) = f(x(t), u(t)) \text{ for almost all } t \geq 0, \\ x(0) = x_0 \end{cases}$$

where $x_0 \in \mathbb{R}^d$, $f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$, U is a compact metric space and the control $u : \mathbb{R}_+ \rightarrow U$ is a Borel function. Later we will make assumptions on f ensuring that for a fixed initial condition x_0 and a fixed control u the system (1) has a unique solution denoted by $x_{u,x_0} : \mathbb{R}_+ \rightarrow \mathbb{R}^d$, or simply by $x(\cdot)$ if there is no risk of confusion.

Let us denote by $\Delta(\mathbb{R}_+)$ the set of Borel probability measures in $\mathbb{R}_+$ and let $\theta \in \Delta(\mathbb{R}_+)$. Then we associate to any $x_0 \in \mathbb{R}^d$ the following optimization problem

$$(2) \quad V_\theta(x_0) = \inf_{u \in \mathcal{U}} \int_{\mathbb{R}_+} \ell(x_{u,x_0}(t), u(t)) d\theta(t),$$

where $\mathcal{U}$ denotes the set of Borel measurable controls $u : \mathbb{R}_+ \rightarrow U$ and $\ell : \mathbb{R}^d \times U \rightarrow \mathbb{R}$ denotes the running cost, which is a nonnegative Borel measurable bounded function.

Particular cases of such value functions V_θ are:

- Value defined through a Cesaro mean $d\theta(t) = \frac{1}{T} \mathbb{1}_{[0,T]}(t) dt$ (with $T > 0$), namely:

$$V_\theta(x_0) = \inf_{u \in \mathcal{U}} \frac{1}{T} \int_0^T \ell(x_{u,x_0}(t), u(t)) dt$$

- Value defined through an Abel mean $d\theta(t) = \lambda e^{-\lambda t} dt$, namely:

$$V_\theta(x_0) := \inf_{u \in \mathcal{U}} \int_0^{+\infty} \ell(x_{u,x_0}(t), u(t)) \lambda e^{-\lambda t} dt.$$

The limits of the above value-functions when the parameter T tends to $+\infty$ or λ tends to 0^+ have been extensively studied in the literature of control theory [1–7, 10–14, 17, 18, 21]. Many works are devoted to the so-called *ergodic case* namely when the limits exist and are independent of the initial condition x_0 [1, 2, 9]. The relations between the two limits have been studied in [13, 14, 17] and also in [3, 15] where it is shown that there is only one possible limit - in uniform convergence topology - which can be expressed by a representation formula. Usually in the literature the fact that the existence of the limit of Cesaro values implies the existence of the limit of Abel values is known as an Abelian theorem (while the converse statement is called a Tauberian theorem).

The limits $T \rightarrow +\infty$ and $\lambda \rightarrow 0^+$ in the Cesaro and Abel averages, respectively, reflect our interest in the long-term behavior of the averaging process. For a sequence of probability measures $\{\theta_k\}_k$ a long-run phenomenon means, roughly speaking, that for any time $S > 0$, the mass assigned by θ_k to $[0, S]$ tends to zero as $k \rightarrow \infty$.

This paper concerns the asymptotic behavior of values V_{θ_k} defined along a sequence of generalized means $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$. This problem has been studied for discrete time dynamics [16, 19, 20, 22] and also in [6]. In [15], a more precise *Long-Term Condition* is introduced for a sequence $\{\theta_k\}_k$ to characterize its long-run behavior. The authors also establish a representation formula for the limit, provided that the limit exists with respect to the topology of uniform convergence.

In this work, our goal is to obtain a general Abelian-type result, that is, assuming that the limit of Cesaro values exists, we endeavor to establish, under adequate assumptions, the existence of the limit of values obtained by a generalized sequence of probability measures $\{\theta_k\}_k$ and prove that the limits coincide.

If no ergodicity condition is assumed, the limit value (in case it exists) is not constant, but depends on the initial condition. In the relevant literature, ergodicity is often obtained through some dissipativity condition, controllability assumptions or coercivity conditions on the Hamiltonian with respect to the adjoint variable [9]. In this work, we will not make any of these assumptions and treat directly the general case, through a control of the asymptotic behavior of the sequence $\{\theta_k\}_k$. This will eventually ensure the validity of a generalized Abelian theorem, where the value of the problem will be a function of the initial state.

As a byproduct of our result we show that without loss of generality we can restrict to probability measures $\{\theta_k\}_k$ that are absolutely continuous with respect to Lebesgue measure and have smooth densities.

The paper is organized as follows: Section 2 is devoted to preliminary results and notation. Section 3 studies properties of a sequence of probability measures satisfying the so-called Long Term Condition guaranteeing good asymptotic properties. The last section deals with the statement and proofs of generalized Abelian results.

2 Preliminaries

Throughout the article we will make the following assumptions:

$$(3) \quad \left\{ \begin{array}{l} \text{the function } f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d \text{ is Borel measurable ;} \\ \quad - f(\cdot, u) \text{ is uniformly Lipschitz, that is, there exists } L > 0 \text{ such that:} \\ \quad \quad \|f(x, u) - f(x', u)\| \leq L \|x - x'\|, \text{ for all } u \in U \text{ and all } (x, x') \in \mathbb{R}^d \times \mathbb{R}^d ; \\ \quad - \text{there exists } \kappa > 0 \text{ such that:} \\ \quad \quad \|f(x, u)\| \leq \kappa(1 + \|x\|) \text{ for all } x \in \mathbb{R}^d; \\ \text{the function } \ell : \mathbb{R}^d \times U \rightarrow \mathbb{R} \text{ is Borel measurable and bounded.} \end{array} \right.$$

Under these assumptions, given any control $u \in \mathcal{U}$ and initial condition $x_0 \in \mathbb{R}^d$, there exists a unique absolutely continuous $t \rightarrow x_{u, x_0}(t)$ to the control system (1) which is defined on $\mathbb{R}_+$. Since the Lagrangian ℓ is bounded, we can assume without loss of generality that the function ℓ takes

values in $[0, 1]$, that is,

$$0 \leq \ell(x, u) \leq 1, \quad \text{for all } x \in \mathbb{R}^d \text{ and } u \in U.$$

It follows easily that for any probability measure $\theta \in \Delta(\mathbb{R}_+)$ and initial condition $x_0 \in \mathbb{R}^d$, the value $V_\theta(x_0)$ is well-defined by formula (2) and

$$0 \leq V_\theta(x_0) \leq 1.$$

We now introduce some notation and definitions for measures.

The Lebesgue measure in $\mathbb{R}_+$ is denoted by $\mathcal{L}$. Let $\theta \in \Delta(\mathbb{R}_+)$. For a set $A \in \mathcal{B}(\mathbb{R}_+)$, its Lebesgue measure is denoted by $|A|$ or $\mathcal{L}(A)$. The Lebesgue decomposition of a measure θ is defined as follows: $\theta = \theta^a + \theta^\sigma$ where the θ^a is the absolutely continuous part (with respect to the Lebesgue measure), that is, $\theta^a \ll \mathcal{L}$ and the singular part θ^σ satisfies $\theta^\sigma \perp \mathcal{L}$. For any signed Borel measure ν on $\mathbb{R}_+$ we denote by ν^+ and ν^- the positive and negative part of ν (that is, $\nu = \nu^+ - \nu^-$) and by $\|\nu\|_{TV} := \nu^+(\mathbb{R}_+) + \nu^-(\mathbb{R}_+)$ the total variation of ν (which is a norm in the space of signed Borel measures). Recall that if ν is a (positive) measure on $\mathbb{R}_+$, then $\|\nu\|_{TV} = \nu(\text{supp}(\nu)) = \nu(\mathbb{R}_+)$ where $\text{supp}(\nu)$ denotes the support of ν .

For any $s > 0$ and $\theta \in \Delta(\mathbb{R}_+)$ we define

$$TV_s(\theta) := \sup_{Q \in \mathcal{B}(\mathbb{R}_+)} |\theta(Q + s) - \theta(Q)|,$$

where $\mathcal{B}(\mathbb{R}_+)$ stands for the family of Borel subsets of $\mathbb{R}_+$. Also, for $S > 0$, we define

$$\overline{TV}_S(\theta) := \sup_{s \in [0, S]} TV_s(\theta).$$

Thanks to the triangle inequality we have

$$TV_{a+b}(\theta) \leq TV_a(\theta) + TV_b(\theta) \quad \text{and} \quad \overline{TV}_{a+b}(\theta) \leq \overline{TV}_a(\theta) + \overline{TV}_b(\theta), \quad \text{for any } a, b \geq 0.$$

Furthermore, an easy computation gives that

$$(4) \quad TV_s(\theta) \leq \|\theta - \tau_s \# \theta\|_{TV} \leq 2TV_s(\theta),$$

where τ_s is the translation by s on $\mathbb{R}$ and $\tau_s \# \theta(A) = \theta(A + s)$, for all $A \in \mathcal{B}(\mathbb{R})$. We also recall that if $\theta \in \Delta(\mathbb{R}_+)$ has a density f_θ (with respect to Lebesgue measure), then we deduce that for every $s \geq 0$

$$TV_s(\theta) \leq \int_{\mathbb{R}_+} |f_\theta(t + s) - f_\theta(t)| dt \leq 2TV_s(\theta).$$

3 Long Term Condition

We now recover an asymptotic condition expressing the large duration of the sequence of means $\{\theta_k\}_k$, introduced in [15, Definition 1.2].

Definition 3.1 (LTC condition). *A sequence $\{\theta_k\}_k \in \Delta(\mathbb{R}_+)$ of probability measures on $\mathbb{R}_+$ satisfies the long term condition (in short, LTC) if*

$$\lim_{k \rightarrow +\infty} \overline{TV}_1(\theta_k) = 0.$$

Note that $\{\theta_k\}_k$ satisfies the (LTC) if and only if, for any $R > 0$,

$$\lim_{k \rightarrow +\infty} \overline{TV}_R(\theta_k) = 0.$$

One can easily check that both sequences $d\theta(t) = \frac{1}{T_k} \mathbb{1}_{[0, T_k]}(t) dt$ (with $T_k \rightarrow +\infty$) of Cesaro means and $d\theta(t) = \lambda e^{-\lambda t} dt$ (when $\lambda_k \rightarrow 0^+$) of Abel means satisfy the LTC condition. We also refer the reader to [15] for several examples of sequences of means satisfying LTC.

One useful feature of the condition LTC is the fact that for any sequence $\{\theta_k\}_k$ satisfying LTC, the sequence V_{θ_k} has at most one possible accumulation point - in uniform convergence topology - and therefore this accumulation point does not depend on the particular choice of the sequence $\{\theta_k\}_k$ satisfying LTC. This follows from [15, Theorem 4.1]. An important consequence is the existence of a limit - in uniform convergence topology - when the sequence of functions is equicontinuous. A byproduct of this result is an Abelian-Tauberian theorem.

In order to obtain our main results, we need to analyze the LTC condition.

The first result shows that for a sequence of means satisfying LTC, the sequence of the total variations of the singular parts of the means vanishes

Lemma 3.2. *Let $\theta \in \Delta(\mathbb{R}_+)$. Then $TV_s(\theta) \geq \|\theta^\sigma\|_{TV}$ for almost every $s \in [0, 1]$.*

In particular, if $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ satisfies LTC, we have that $\|\theta_k^\sigma\|_{TV} \leq \overline{TV}_1(\theta_k) \xrightarrow[k \rightarrow \infty]{} 0$.

Proof. Let $\theta \in \Delta(\mathbb{R}_+)$ and consider its Lebesgue decomposition $\theta = \theta^a + \theta^\sigma$. Let $N \subset \mathbb{R}_+$ be a Lebesgue-null set such that $\theta^\sigma(\mathbb{R}_+ \setminus N) = 0$. Then $\mathcal{L}(N) = \mathcal{L}(-N + t) = 0$, for all $t \geq 0$. Thanks to Fubini's theorem, we have that

$$\underbrace{\int_0^1 \int_0^{+\infty} \mathbb{1}_{N+s}(t) d\theta(t) ds}_{\theta(N+s)} = \int_0^{+\infty} \underbrace{\int_0^1 \mathbb{1}_{N+s}(t) d\lambda(s)}_{\leq \mathcal{L}(-N+t) = \mathcal{L}(N) = 0} d\theta(t) = \int_0^{+\infty} 0 d\theta(t) = 0.$$

Therefore, $\theta(N + s) = 0$, for almost every $s \in [0, 1]$. Therefore,

$$(5) \quad TV_s(\theta) \geq \theta(N) - \theta(N + s) = \|\theta^\sigma\|_{TV}, \quad \text{for a.e. } s \in [0, 1].$$

Let now $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$. Then, for any $k \geq 1$ there exists a null set A_k such that for all $s \in [0, 1] \setminus A_k$ we have: $TV_s(\theta_k) \geq \|\theta_k^\sigma\|_{TV}$. Setting $A = \bigcup_{k \geq 1} A_k$, and taking $s \in [0, 1] \setminus A$, we readily have

$$\|\theta_k^\sigma\|_{TV} \leq TV_s(\theta_k) \leq \overline{TV}_1(\theta_k),$$

and evoking LTC the result follows. $\square$

We will now provide an equivalent formulation of the LTC.

Proposition 3.3. *Let $\{\theta_k\}_k \in \Delta(\mathbb{R}_+)$ be a sequence of probability measures on $\mathbb{R}_+$. The following assertions are equivalent.*

- a) $\{\theta_k\}_k$ satisfies the LTC.
- b) There is $R > 0$ such that, for all $s \in [0, R]$, $\lim_{k \rightarrow +\infty} TV_s(\theta_k) = 0$.

c) For all $s \in [0, 1]$, $\lim_{k \rightarrow +\infty} TV_s(\theta_k) = 0$.

The proof of Proposition 3.3 requires the following technical result.

Lemma 3.4. *Let $\theta \in \Delta(\mathbb{R}_+)$ be a probability measure which is absolutely continuous with respect to $\mathcal{L}$. Then, the map $s \mapsto TV_s(\theta)$ is lower semicontinuous.*

Proof. Let $f := \frac{\partial \theta}{\partial \lambda}$ be the density of θ . Let $s \in [0, 1]$. By regularity of the θ , for any $Q \in \mathcal{B}(\mathbb{R}_+)$ and any $\varepsilon > 0$, there is an open set $U \subset \mathbb{R}_+$ such that

$$|\theta(Q) - \theta(Q + s) - \theta(U) + \theta(U + s)| \leq \varepsilon.$$

Moreover, U being open it is a disjoint union of countable many open intervals which can be approximated - with respect to the signed measure $\theta - \tau_s \# \theta$ - by a set which is finite disjoint union of intervals of the form $W := \bigcup_{n=1}^k (a_n, b_n)$. So,

$$TV_s(\theta) = \sup \left\{ |\theta(W) - \theta(W + s)| : W = \bigcup_{n=1}^k (a_n, b_n) \subset \mathbb{R}_+ \right\}.$$

Let $\varepsilon > 0$ and take $W = \bigcup_{n=1}^k (a_n, b_n)$ such that

$$TV_s(\theta) \leq |\theta(W) - \theta(W + s)| + \varepsilon = \left| \int_{\mathbb{R}_+} f(x) \mathbb{1}_W dx - \int_{\mathbb{R}_+} f(x) \mathbb{1}_{W+s}(x) dx \right| + \varepsilon.$$

Since W is a finite union of open intervals, we have that

$$f(x) \mathbb{1}_{W+t}(x) \xrightarrow[t \rightarrow s]{} f(x) \mathbb{1}_{W+s}(x), \quad \text{for almost every } x \in \mathbb{R}_+.$$

Applying the dominated convergence theorem ($f \in \mathcal{L}^1$), we obtain

$$TV_s(\theta) \leq \lim_{t \rightarrow s} \left| \int f(x) \mathbb{1}_W(x) dx - \int f(x) \mathbb{1}_{W+t}(x) dx \right| + \varepsilon \leq \liminf_{t \rightarrow s} TV_t(\theta) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can send it to 0 and finally conclude that

$$TV_s(\theta) \leq \liminf_{t \rightarrow s} TV_t(\theta).$$

This completes the proof. □

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. a) $\implies$ b) : There is nothing to do.

b) $\implies$ c) : Let $s \in [0, 1]$, and choose $m \in \mathbb{N}$ such that $s/m \leq R$. By the subadditivity of $s \mapsto TV_s(\theta_k)$, we have

$$TV_s(\theta_k) \leq m TV_{s/m}(\theta_k) \xrightarrow[k \rightarrow +\infty]{} 0.$$

Thus, c) holds.

c) $\implies$ a) : Thanks to Lemma 3.2, we know that $\|\theta_k^\sigma\|_{TV} \rightarrow 0$. Set

$$\alpha_k := \theta_k^a(\mathbb{R}_+) \quad \text{and} \quad \rho_k := \frac{\theta_k^a}{\alpha_k}$$

for all sufficiently large k . Since $\alpha_k \rightarrow 1$, we have

$$\|\rho_k - \theta_k\|_{TV} \leq \|\theta_k^\sigma\|_{TV} + \|\theta_k^a\|_{TV} \left| \frac{1}{\alpha_k} - 1 \right| \longrightarrow 0.$$

Moreover, for every $s \geq 0$,

$$|TV_s(\rho_k) - TV_s(\theta_k)| \leq 2\|\rho_k - \theta_k\|_{TV}.$$

Consequently, replacing θ_k by ρ_k affects neither condition c) nor the desired conclusion. Thus, without loss of generality, we can assume that, for all $k \in \mathbb{N}$, θ_k is absolutely continuous with respect to $\mathcal{L}$.

We show first that, for any $r \in (0, 1]$, we have

$$\lim_{k \rightarrow +\infty} \sup_{s \in [r, 1]} TV_s(\theta_k) = 0.$$

Fix $r > 0$ and assume towards a contradiction that, up to a subsequence, there exist $\varepsilon > 0$ and $\{s_k\}_k \subset [r, 1]$ such that

$$TV_{s_k}(\theta_k) > \varepsilon.$$

Up to a new subsequence, we can and shall assume that $\{s_k\}_k$ converges to $\bar{s} \in [r, 1]$.

Observe now that, for any $a, b \geq 0$ such that $a + b = s_k$, we have

$$\varepsilon < TV_{s_k}(\theta_k) \leq TV_a(\theta_k) + TV_b(\theta_k).$$

Consider the sets

$$A_k := \left\{ a \in [0, 2^{-1}s_k] : TV_a(\theta_k) > \frac{\varepsilon}{2} \right\},$$

$$B_k := \left\{ b \in [2^{-1}s_k, s_k] : TV_b(\theta_k) > \frac{\varepsilon}{2} \right\}.$$

Note that

$$(s_k - A_k) \cup B_k = [2^{-1}s_k, s_k].$$

From Lemma 3.4, we know that $s \mapsto TV_s(\theta_k)$ is lower semicontinuous, so the sets A_k and B_k are open subsets of $[0, 2^{-1}s_k]$ and $[2^{-1}s_k, s_k]$, respectively.

Let $t, \rho > 0$ be such that

$$[t - \rho, t + \rho] \subset (2^{-1}\bar{s}, \bar{s}).$$

Claim: $B_k \cap [t - \rho, t + \rho] \neq \emptyset$ for infinitely many $k \in \mathbb{N}$.

Indeed, otherwise, there is an infinite set $\mathbb{M} \subset \mathbb{N}$ such that

$$B_k \cap [t - \rho, t + \rho] = \emptyset, \quad \text{for all } k \in \mathbb{M}.$$

For all sufficiently large $k \in \mathbb{M}$, we have

$$[t - \rho, t + \rho] \subset [2^{-1}s_k, s_k].$$

Since

$$[2^{-1}s_k, s_k] = (s_k - A_k) \cup B_k,$$

we deduce that

$$[s_k - t - \rho, s_k - t + \rho] \subset A_k, \quad \text{for all sufficiently large } k \in \mathbb{M}.$$

Set $\bar{a} := \bar{s} - t$. Since $s_k \rightarrow \bar{s}$, we have

$$\bar{a} \in [s_k - t - \rho, s_k - t + \rho]$$

for all sufficiently large $k \in \mathbb{M}$. Hence,

$$TV_{\bar{a}}(\theta_k) > \frac{\varepsilon}{2}$$

for infinitely many k , which contradicts the fact that $TV_{\bar{a}}(\theta_k) \rightarrow 0$. The claim is proved.

Claim: There is an infinite set $\mathbb{M}$ such that $\bigcap_{k \in \mathbb{M}} B_k \neq \emptyset$.

Let $k_1 \in \mathbb{N}$ be sufficiently large and such that

$$B_{k_1} \cap [t - \rho, t + \rho] \neq \emptyset.$$

Since B_{k_1} is open in $[2^{-1}s_{k_1}, s_{k_1}]$, there exist $t_1, r_1 > 0$ such that

$$[t_1 - r_1, t_1 + r_1] \subset \text{int } B_{k_1} \cap [t - \rho, t + \rho].$$

Applying the first claim, there exists $k_2 > k_1$ such that

$$B_{k_2} \cap [t_1 - r_1, t_1 + r_1] \neq \emptyset.$$

Since B_{k_2} is open in $[2^{-1}s_{k_2}, s_{k_2}]$, there exist $t_2, r_2 > 0$ such that

$$[t_2 - r_2, t_2 + r_2] \subset (t_1 - r_1, t_1 + r_1) \cap \text{int } B_{k_2}, \quad r_2 < 2^{-1}r_1.$$

Inductively, we generate sequences $\{t_n\}_n, \{r_n\}_n \subset \mathbb{R}_+$ and an increasing sequence $\{k_n\}_n$ such that $r_n \rightarrow 0$ and

$$[t_{n+1} - r_{n+1}, t_{n+1} + r_{n+1}] \subset (t_n - r_n, t_n + r_n) \cap \text{int } B_{k_{n+1}}.$$

The nested intervals therefore have a unique common point $\bar{b}$, and

$$\bar{b} \in \bigcap_{n \in \mathbb{N}} B_{k_n}.$$

The proof of the claim is complete.

Let $\mathbb{M} := \{k_n : n \in \mathbb{N}\}$ be provided by the second claim. Then, for every

$$\bar{b} \in \bigcap_{k \in \mathbb{M}} B_k,$$

we have

$$TV_{\bar{b}}(\theta_k) > \frac{\varepsilon}{2}, \quad \text{for all } k \in \mathbb{M}.$$

This contradicts the fact that $TV_{\bar{b}}(\theta_k) \rightarrow 0$.

So far, we have proved that, for any $r \in (0, 1]$,

$$\lim_{k \rightarrow \infty} \sup_{s \in [r, 1]} TV_s(\theta_k) = 0.$$

Fix $r = 1/3$ and consider $s \in (0, r)$. Then, for any $Q \in \mathcal{B}(\mathbb{R}_+)$, we have

$$\begin{aligned} |\theta_k(Q + s) - \theta_k(Q)| &\leq |\theta_k(Q + s + r) - \theta_k(Q + s)| + |\theta_k(Q + s + r) - \theta_k(Q)| \\ &\leq TV_r(\theta_k) + TV_{s+r}(\theta_k) \leq 2 \sup_{\tau \in [r, 1]} TV_\tau(\theta_k). \end{aligned}$$

By taking the supremum over $Q \in \mathcal{B}(\mathbb{R}_+)$, we deduce that

$$TV_s(\theta_k) \leq 2 \sup_{\tau \in [r, 1]} TV_\tau(\theta_k), \quad \text{for all } s \in (0, r).$$

The same inequality is immediate for $s \in [r, 1]$, while $TV_0(\theta_k)$ is always 0. Therefore,

$$\overline{TV}_1(\theta_k) \leq 2 \sup_{\tau \in [r, 1]} TV_\tau(\theta_k).$$

It follows that

$$\lim_{k \rightarrow +\infty} \overline{TV}_1(\theta_k) = 0.$$

This completes the proof. $\square$

Remark 3.5. The above proposition provides a negative answer to [15, Question 1].

In what follows, we present some properties that reinforce the intuition that if $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ satisfies LTC, then asymptotically the mass of θ_k goes to infinity. Indeed, as an immediate consequence of the LTC, for any $T > 0$ we have

$$(6) \quad \theta_k([0, T]) \leq TV_T(\theta_k) \longrightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

Proposition 3.6. *Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be a sequence satisfying LTC. Then, for any $N \geq 0$ and any sequence $A_k \subset \mathcal{B}(\mathbb{R}_+)$, with $|A_k| = N$, we have*

$$\lim_{k \rightarrow \infty} \theta_k(A_k) = 0.$$

In other words, for any $N \geq 0$, the sequence $M_k^N := \sup\{\theta_k(A) : A \in \mathcal{B}(\mathbb{R}_+), |A| \leq N\}$ tends to 0 as k tends to infinity.

Proof. Reasoning towards a contradiction, let us assume that there exist $N, \delta > 0$, and a sequence of sets $\{A_k\}_k \subset \mathcal{B}(\mathbb{R}_+)$ such that (up to passing to a subsequence, still denoted in the same way)

$$\theta_k(A_k) \geq \delta, \quad \text{for all } k \in \mathbb{N}.$$

Let $M > \frac{N}{\delta}$. Since $\{\theta_k\}_k$ satisfies the LTC property, we have $\overline{TV}_M(\theta_k)$ tends to 0 as k tends to $+\infty$. Moreover, since A_k participates in the supremum defining $TV_s(\theta_k)$, we deduce that $|\theta_k(A_k) - \theta_k(A_k + s)| \leq \overline{TV}_M(\theta_k)$ for all $s \in [0, M]$. So

$$\theta_k(A_k + s) \geq \theta_k(A_k) - \overline{TV}_M(\theta_k) \geq \delta - \overline{TV}_M(\theta_k).$$

We integrate the above inequality on $s \in [0, M]$ to get

$$\begin{aligned} M(\delta - \overline{TV}_M(\theta_k)) &\leq \int_0^{+\infty} \mathbb{1}_{[0, M]}(s) \int_0^{+\infty} \mathbb{1}_{A_k+s}(t) d\theta_k(t) ds \\ &= \int_0^{+\infty} \int_0^{+\infty} \mathbb{1}_{[0, M] \cap (t - A_k)}(s) ds d\theta_k(t) \\ &\leq \int_0^{+\infty} |A_k| d\theta_k(t) = N. \end{aligned}$$

After sending k to $+\infty$ we deduce $M\delta \leq N$, which is a contradiction with the choice of M . $\square$

The following property catches the idea that if a sequence satisfies the LTC condition, then it should asymptotically distribute as a uniform measure at infinity. For every $P > 0$, we write

$$\mathcal{L}_P := \frac{1}{P} \mathcal{L}|_{[0, P)} \in \Delta([0, P)).$$

Proposition 3.7. *Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be a sequence and let $P > 0$. Define $\nu_k^P \in \Delta([0, P))$ by*

$$(7) \quad \nu_k^P(A) := \theta_k(A + P\mathbb{N}) = \theta \left(\bigcup_{n=0}^{\infty} A + nP \right), \quad \text{for all } A \in \mathcal{B}([0, P)).$$

Assume that $\{\theta_k\}_k$ satisfies the LTC. Then,

$$\lim_{k \rightarrow +\infty} \|\nu_k^P - \mathcal{L}_P\|_{TV} = 0,$$

where the above limit is uniform on bounded sets with respect to $P \in (0, +\infty)$.

Proof. Let $\theta \in \Delta(\mathbb{R}_+)$ and let $\nu \in \Delta([0, P))$ be defined as in (7) using θ .

For $s \in [0, P)$, we denote by $\tau_s \# \nu \in \Delta([0, P))$ the measure defined as the translation of ν by s modulo P . That is,

$$\tau_s \# \nu(A) = \nu((A + s) \cap [0, P)) + \nu((A + s - P) \cap [0, P)), \quad \text{for all } A \in \mathcal{B}([0, P)).$$

It follows that

$$\mathcal{L}_P(A) = \int_{[0, P)} \tau_s \# \nu(A) d\mathcal{L}_P(s), \quad \text{for all } A \in \mathcal{B}([0, P)).$$

On the other hand, it easily follows that, for every $A \in \mathcal{B}([0, P])$,

$$|\nu(A) - \tau_s \sharp \nu(A)| \leq TV_s(\theta) + \theta([0, P]) \leq 2\overline{TV}_P(\theta).$$

Since ν and $\tau_s \sharp \nu$ are probability measures, we have

$$\|\nu - \tau_s \sharp \nu\|_{TV} = 2 \sup_{A \in \mathcal{B}([0, P])} |\nu(A) - \tau_s \sharp \nu(A)| \leq 4\overline{TV}_P(\theta), \quad \text{for all } s \in [0, P].$$

Let now $\{\theta_k\}_k$ and $\{\nu_k^P\}_k$ be as in the statement. Then, for any $A \in \mathcal{B}([0, P])$,

$$|\nu_k^P(A) - \mathcal{L}_P(A)| \leq \frac{1}{P} \int_{[0, P]} |\nu_k^P(A) - \tau_s \sharp \nu_k^P(A)| ds \leq 2\overline{TV}_P(\theta_k).$$

Since ν_k^P and $\mathcal{L}_P$ are probability measures, it follows that

$$\|\nu_k^P - \mathcal{L}_P\|_{TV} \leq 4\overline{TV}_P(\theta_k).$$

The LTC condition therefore yields

$$(8) \quad \lim_{k \rightarrow +\infty} \|\nu_k^P - \mathcal{L}_P\|_{TV} = 0.$$

Let us finally observe that for any $P_0 > 0$, we have

$$\overline{TV}_P(\theta_k) \leq \overline{TV}_{P_0}(\theta_k) \xrightarrow[k \rightarrow +\infty]{} 0, \quad \text{for all } P \in [0, P_0].$$

Thus, the limit (8) is uniform on bounded sets with respect to P . □

We finish this section with two results that will be useful in the forthcoming section.

Lemma 3.8. *Let $\theta \in \Delta(\mathbb{R}_+)$ be a probability measure absolutely continuous with respect to $\mathcal{L}$ and let $S > 0$. Define the measure $\nu(\theta, S)$ on $\mathbb{R}_+$, absolutely continuous with respect to $\mathcal{L}$, by*

$$\frac{\partial \nu(\theta, S)}{\partial \mathcal{L}}(s) := \frac{1}{S} \theta([\max(0, s - S), s]).$$

Then, $\nu(\theta, S) \in \Delta(\mathbb{R}_+)$ and

$$\sup_{Q \in \mathcal{B}(\mathbb{R}_+)} |\theta(Q) - \nu(\theta, S)(Q)| \leq 3\overline{TV}_S(\theta).$$

Proof. It immediately follows that $\nu := \nu(\theta, S)$ is a positive Borel measure. Let us first show that $\nu \in \Delta(\mathbb{R}_+)$. Indeed,

$$\begin{aligned} \nu(\mathbb{R}_+) &= \int_0^\infty \frac{1}{S} \theta([\max(0, s - S), s]) ds = \int_0^\infty \int_{\max(0, s - S)}^s \frac{1}{S} d\theta(t) ds \\ &= \int_0^\infty \int_t^{t+S} \frac{1}{S} ds d\theta(t) = 1. \end{aligned}$$

where the third equality follows by Fubini's theorem. Noticing that

$$\theta([0, S]) = \theta(\mathbb{R}_+) - \theta(S + \mathbb{R}_+) \leq TV_S(\theta),$$

we deduce that

$$\nu(\theta, S)([0, S]) = \int_0^S \frac{\theta([0, s])}{S} ds \leq TV_S(\theta).$$

Let now $Q \in \mathcal{B}(\mathbb{R}_+)$ be such that $Q \cap [0, S] = \emptyset$. We can compute

$$\begin{aligned} \nu(\theta, S)(Q) &= \int_Q \frac{\theta([s-S, s])}{S} ds = \int_0^\infty \int_0^\infty \frac{1}{S} \mathbb{1}_Q(s) \mathbb{1}_{[s-S, s]}(t) d\theta(t) ds \\ &= \int_0^\infty \frac{1}{S} \mathcal{L}((t, t+S] \cap Q) d\theta(t) \end{aligned}$$

On the other hand, denoting by $\tilde{Q} := Q - S$ we can compute

$$\begin{aligned} \int_0^S \frac{1}{S} \theta(\tilde{Q} + s) ds &= \int_0^\infty \int_0^\infty \frac{1}{S} \mathbb{1}_{(0, S]}(s) \mathbb{1}_{\tilde{Q}+s}(t) d\theta(t) ds \\ &= \int_0^\infty \frac{1}{S} \mathcal{L}((0, S] \cap (t - \tilde{Q})) d\theta(t) \\ &= \int_0^\infty \frac{1}{S} \mathcal{L}([t, t+S] \cap Q) d\theta(t) \\ &= \nu(\theta, S)(Q), \end{aligned}$$

where in the second line we have used Fubini's theorem and in the third one the invariance under translation and rotation of the Lebesgue measure. Now, noting that for $s \in [0, S]$ we have that

$$|\theta(Q) - \theta(\tilde{Q} + s)| = |\theta(Q) - \theta(Q - S + s)| \leq TV_{S-s}(\theta) \leq \overline{TV}_S(\theta).$$

Hence

$$|\theta(Q) - \nu(\theta, S)(Q)| \leq \overline{TV}_S(\theta).$$

Finally, for an arbitrary Borel set $Q \in \mathcal{B}(\mathbb{R}_+)$ we have

$$\begin{aligned} |\theta(Q) - \nu(\theta, S)(Q)| &\leq \theta([0, S]) + \nu(\theta, S)([0, S]) + |\theta(Q \cap [S, +\infty)) - \nu(\theta, S)(Q \cap [S, +\infty))| \\ &\leq 3\overline{TV}_S(\theta). \end{aligned}$$

This completes the proof. $\square$

The following proposition will be used to simplify the setting of the next section. It essentially asserts that, when we are interested in the asymptotic behavior of the value functions V_{θ_k} , we can limit our attention to sequences of absolutely continuous probability measures with smooth densities.

Proposition 3.9. *Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be a sequence of measures that satisfies LTC. Then, there exists a sequence $\{\tilde{\theta}_k\}_k \subset \Delta(\mathbb{R}_+)$ of absolutely continuous measures ($\tilde{\theta}_k \ll \mathcal{L}$) with $\frac{\partial \tilde{\theta}_k}{\partial \mathcal{L}} \in C^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$, satisfying LTC and*

$$\lim_{k \rightarrow \infty} \|V_{\theta_k} - V_{\tilde{\theta}_k}\|_\infty = 0$$

Proof. Thanks to Lemma 3.2, the absolutely continuous part of θ_k , that is θ_k^a , satisfies $\|\theta_k^a\|_{TV} \rightarrow 1$. Therefore, for k sufficiently large, we have $\theta_k^a(\mathbb{R}_+) > 0$. Setting $\rho_k := \frac{1}{\theta_k^a(\mathbb{R}_+)}\theta_k^a$, we deduce that the sequence $\{\rho_k\}_k$ satisfies $\|\rho_k - \theta_k\|_{TV} \rightarrow 0$. Hence, thanks to (4), we deduce that $\{\rho_k\}_k$ also satisfies LTC. Note that ρ_k is absolutely continuous with respect to $\mathcal{L}$. It is well known that, for each $k \in \mathbb{N}$, we can find another absolutely continuous measure $\tilde{\theta}_k \in \Delta(\mathbb{R}_+)$ such that $\|\rho_k - \tilde{\theta}_k\|_{TV} < 1/k$, $\frac{\partial \tilde{\theta}_k}{\partial \mathcal{L}}$ is continuous in $\mathbb{R}_+$ and continuously differentiable in $(0, +\infty)$. In particular, $\{\tilde{\theta}_k\}_k$ satisfies LTC. Finally, since $\ell(\mathbb{R}^d \times U) \subset [0, 1]$, for any $u \in \mathcal{U}$ we get

$$\left| \int_0^\infty \ell(x_{x_0, u}(t), u(t)) d\theta_k(t) - \int_0^\infty \ell(x_{x_0, u}(t), u(t)) d\tilde{\theta}_k(t) \right| \leq \|\theta_k - \rho_k\|_{TV} + \|\rho_k - \tilde{\theta}_k\|_{TV}.$$

Thus,

$$\lim_{k \rightarrow +\infty} \|V_{\theta_k} - V_{\tilde{\theta}_k}\|_\infty = 0$$

This completes the proof. $\square$

4 Abelian theorem for general means

When $d\theta = \frac{1}{T}\mathbb{1}_{[0, T]}(s)ds$ and $T > 0$ we denote by a slight abuse of notation $V_T := V_\theta$.

The main problem we address is the following

Question 4.1. *Assume that V_T converges uniformly to V^* as T goes to infinity. Given a sequence $\{\theta_k\}_k$, is it true that V_{θ_k} converges uniformly to the same value V^* ?*

Without additional assumptions on the sequence $\{\theta_k\}_k$, the answer is clearly negative. A simple counterexample is obtained by taking $\theta_k = \delta_1$ for every $k \geq 1$. When $\{\theta_k\}_k$ is the sequence of Abel means, with $\lambda_k \rightarrow 0$, this question was positively solved in [17] or [14] (cf. also [2] for the case where V^* is constant).

To fix ideas, throughout this section we always assume

$$(\mathcal{H}) \quad V_T \rightarrow V^* \in C(\mathbb{R}^d) \text{ uniformly as } T \rightarrow \infty.$$

Remark 4.2. Although the optimal control problem (1)–(2) was stated in $\mathbb{R}^d$, the derived results, as well as the proofs, remain valid when the problem is stated on an *invariant* subset Ω of $\mathbb{R}^d$. We recall that a set Ω is said to be invariant for the control system (1) when for any initial condition $x_0 \in \Omega$ and any control $u \in \mathcal{U}$ the associated solution never escapes from Ω . Namely for all $t \geq 0$, we have $x_{u, x_0}(t) \in \Omega$.

Remark 4.3. For the above question, the required property is the uniform convergence of the values defined by Cesaro means. Several sufficient conditions ensuring this property have been established in the literature. Under suitable controllability conditions and/or coercivity assumptions on the associated Hamiltonian, uniform convergence holds with the limit function V^* being constant. In [18], the uniform convergence of the Cesaro values to a possibly nonconstant V^* is established in the nonexpansive case.

Another natural question concerns the convergence of the values with respect to the topology of pointwise convergence. If only the pointwise convergence of the Cesaro mean values is assumed, convergence of the Abel mean values to the same limit may fail (*cf.* counterexample in [17, Section 4]).

To study the above question our strategy is to consider successively the following problems

$$\liminf V_{\theta_k} \geq V^* \text{ and } \limsup V_{\theta_k} \leq V^*$$

where limsup and liminf should be understood in the uniform topology sense.

4.1 Value functions: controlling the limit inferior of V_{θ_k}

The main result of this section reads as follows.

Theorem 4.4. *Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be a sequence of measures satisfying LTC. Assume that $(\mathcal{H})$ holds. Then*

$$V^* \leq \liminf V_{\theta_k} \quad \text{uniformly,}$$

that is,

$$(9) \quad \liminf_{k \rightarrow \infty} \inf_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \geq 0.$$

Proof. The proof is divided in 4-steps. Let us first fix $x_0 \in \mathbb{R}^d$ and let us denote by $x_{u,x_0}(\cdot)$ the associated solution to (1) where $u \in \mathcal{U}$ is the used control and x_0 is the starting point.

Step 1: $V^*(x_0) \leq V^*(x_{u,x_0}(t))$ for all $u \in \mathcal{U}$ and all $t > 0$.

Indeed, fix $\bar{u} \in \mathcal{U}$ and $t > 0$. We compute

$$\begin{aligned} V^*(x_0) &= \lim_{T \rightarrow \infty} \inf \left\{ \frac{1}{T} \int_0^T \ell(x_{u,x_0}(s), u(s)) ds : u \in \mathcal{U} \right\} \\ &\leq \lim_{T \rightarrow \infty} \inf \left\{ \frac{1}{T} \int_0^T \ell(x_{u,x_0}(s), u(s)) ds : u \in \mathcal{U}, u|_{[0,t]} = \bar{u}|_{[0,t]} \right\} \\ &= \lim_{T \rightarrow \infty} \frac{T-t}{T} \inf \left\{ \frac{1}{T-t} \int_t^T \ell(x_{u,x_0}(s), u(s)) ds : u \in \mathcal{U}, u|_{[0,t]} = \bar{u}|_{[0,t]} \right\} \\ &= V^*(x_{u,x_0}(t)). \end{aligned}$$

Step 2: Consider

$$V_{t,S}(x_0) := \inf \left\{ \frac{1}{S} \int_{[t,t+S)} \ell(x_{u,x_0}(s), u(s)) ds : u \in \mathcal{U} \right\}.$$

Let $\varepsilon > 0$. Then, there is $S_0 > 0$ (independent of x_0) such that

$$V^*(x_0) \leq \varepsilon + V_{t,S}(x_0), \quad \text{for all } S \geq S_0, t > 0.$$

Indeed, since $V^* = \lim_{T \rightarrow +\infty} V_T$ uniformly, there is $S_0 > 0$ such that $\|V^* - V_S\|_\infty \leq \varepsilon$ for all $S > S_0$. Fix $S > S_0$, $t > 0$ and $\bar{u} \in \mathcal{U}$. Then we have

$$\begin{aligned} V^*(x_0) &\leq V^*(x_{\bar{u},x_0}(t)) \leq \varepsilon + V_S(x_{\bar{u},x_0}(t)) \\ &\leq \varepsilon + \inf \left\{ \frac{1}{S} \int_t^{t+S} \ell(x_{u,x_0}(s), u(s)) ds : u \in \mathcal{U}, u|_{[0,t]} = \bar{u}|_{[0,t]} \right\}. \end{aligned}$$

Taking infimum in $\bar{u} \in \mathcal{U}$ we get

$$V^*(x_0) \leq \varepsilon + V_{t,S}(x_0).$$

Step 3: Let $\theta \in \Delta(\mathbb{R}_+)$ be a probability measure, which is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$. Fix $S > 0$ and consider $\nu(\theta, S) \in \Delta(\mathbb{R}_+)$ the probability measure defined in Lemma 3.8. That is, $\nu(\theta, S)$ is an absolutely continuous probability measure with density

$$s \in \mathbb{R}_+ \mapsto \frac{\partial \nu(\theta, S)}{\partial \mathcal{L}}(s) := \frac{1}{S} \theta([\max(0, s-S), s)).$$

Then, $V^*(x_0) \leq \varepsilon + V_{\nu(\theta, S)}(x_0)$.

Indeed, integrating the inequality from Step 2 with respect to θ we obtain

$$V^*(x_0) \leq \varepsilon + \int_{\mathbb{R}_+} V_{t,S}(x_0) d\theta(t)$$

Let $u \in \mathcal{U}$. We have that (thanks to Fubini's theorem)

$$\begin{aligned} V^*(x_0) &\leq \varepsilon + \int_{\mathbb{R}_+} \int_t^{t+S} \frac{1}{S} \ell(x_{u,x_0}(s), u(s)) ds d\theta(t) \\ &= \varepsilon + \int_{\mathbb{R}_+} \int_{\max(0, s-S)}^s \frac{1}{S} \ell(x_{u,x_0}(s), u(s)) d\theta(t) ds \\ &= \varepsilon + \int_{\mathbb{R}_+} \frac{\partial \nu(\theta, S)}{\partial \mathcal{L}}(s) \ell(x_{u,x_0}(s), u(s)) ds \end{aligned}$$

Hence, we have

$$V^*(x_0) \leq \varepsilon + \int_{\mathbb{R}_+} \ell(x_{u,x_0}(s), u(s)) d\nu(\theta, S)(s).$$

Because u is an arbitrary control, after taking infimum on $u \in \mathcal{U}$, we deduce that

$$V^*(x_0) \leq \varepsilon + V_{\nu(\theta, S)}(x_0).$$

Step 4: $|V_\theta(x_0) - V_{\nu(\theta, S)}(x_0)| \leq 3\overline{TV}_S(\theta)$.

Indeed, since $\ell(\mathbb{R}^d \times U) \subset [0, 1]$, for any control $u \in \mathcal{U}$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}_+} \ell(x_{u,x_0}(t), u(t)) d\theta(t) - \int_{\mathbb{R}_+} \ell(x_{u,x_0}(t), u(t)) d\nu(\theta, S)(t) \right| &= \left| \int_{\mathbb{R}_+} \ell(x_{u,x_0}(t), u(t)) d(\theta - \nu(\theta, S))(t) \right| \\ &\leq \sup_{Q \in \mathcal{B}(\mathbb{R}_+)} |\theta(Q) - \nu(\theta, S)(Q)| \end{aligned}$$

Applying now Lemma 3.8, for any control $u \in \mathcal{U}$ we have

$$\left| \int_{\mathbb{R}_+} \ell(x_{u,x_0}(t), u(t)) d\theta(t) - \int_{\mathbb{R}_+} \ell(x_{u,x_0}(t), u(t)) d\nu(\theta, S)(t) \right| \leq 3\overline{TV}_S(\theta).$$

From above, we immediately conclude that $|V_\theta(x_0) - V_{\nu(\theta,S)}(x_0)| \leq 3\overline{TV}_S(\theta)$.

Conclusion: Let $\{\theta_k\}_k$ be a sequence that satisfies LTC and let $\varepsilon > 0$. Thanks to Proposition 3.9, we may assume that the measures $\{\theta_k\}_k$ are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$. Consider $S_0 > 0$ as in Step 2. Then, from Steps 3 and 4 we get for any x_0

$$0 \leq V_{\theta_k}(x_0) - V^*(x_0) + \varepsilon + 3\overline{TV}_{S_0}(\theta_k), \quad \text{for all } k \in \mathbb{N}.$$

Taking the infimum over $x_0 \in \mathbb{R}^d$, subsequently the limit inferior as k goes ∞ and finally sending ε to 0, we obtained the desired conclusion. $\square$

4.2 Value functions: controlling the limit superior of V_{θ_k}

Our goal now is to find sufficient conditions that ensure

$$(10) \quad V^*(x_0) \geq \limsup_{k \rightarrow \infty} V_{\theta_k}(x_0),$$

where the above limit superior can be taken uniformly on the domain of V^* . Observe that these conditions, combined with the LTC condition on the sequence $\{\theta_k\}_k$, will provide a generalized Abelian theorem, namely, the uniform convergence of V_{θ_k} towards V^* .

Thanks to Proposition 3.9, without loss of generality we consider a sequence $\{\theta_k\}_k$ in $\Delta(\mathbb{R}_+)$, such that $\theta_k \ll \mathcal{L}$ (the Lebesgue measure in $\mathbb{R}_+$) for every $k \geq 1$, and the Radon-Nikodym derivative is continuous on $\mathbb{R}_+$ and C^1 -smooth function, that is,

$$(11) \quad f_{\theta_k} := \frac{\partial \theta_k}{\partial \mathcal{L}} \in C^1(\mathbb{R}_+) \cap C(\mathbb{R}_+).$$

We associate to $\{\theta_k\}_k$ the following sequence of (formal) signed measures $\{\mu_k\}_k$ defined, for $k \geq 1$, through their densities

$$\mu'_k(t) := \frac{\partial \mu_k}{\partial \mathcal{L}}(t) := -t f'_{\theta_k}(t), \quad \text{for } t > 0.$$

To ensure that $\{\mu_k\}_k$ is a bounded sequence of real signed measure, we assume the following uniform control on their total variations:

$$(12) \quad M := \sup_{k \in \mathbb{N}} \|\mu_k\|_{TV} < +\infty, \quad \text{where} \quad \|\mu_k\|_{TV} := \int_{\mathbb{R}_+} t |f'_{\theta_k}(t)| dt \quad (\text{total variation of } \mu_k)$$

Observe that in the Abelian case, i.e. $f_\theta(t) = \lambda e^{-\lambda t}$, we have that μ is a probability measure. Therefore, $\|\mu\|_{TV} = 1$.

Notice that $(tf_{\theta_k}(t))' = f_{\theta_k}(t) + tf_{\theta_k}'(t) \in L^1(\mathbb{R}_+)$, whence the function $t \mapsto tf_{\theta_k}(t)$ has a finite limit as $t \rightarrow +\infty$. Since f_{θ_k} is nonnegative and integrable, this limit must be zero. Hence

$$(13) \quad \lim_{t \rightarrow \infty} tf_{\theta_k}(t) = 0.$$

Integrating μ_k' on $[0, T]$ and using integration by parts, we obtain

$$\mu_k([0, T]) = \int_0^T -tf_{\theta_k}'(t) dt = -Tf_{\theta_k}(T) + \int_0^T f_{\theta_k}(t) dt.$$

Taking $T \rightarrow +\infty$, we deduce that

$$\mu_k(\mathbb{R}_+) = 1.$$

Denote by μ_k^+ , μ_k^- the Jordan decomposition of μ_k . Therefore $\mu_k = \mu_k^+ - \mu_k^-$ and for all $Q \in \mathcal{B}(\mathbb{R}_+)$

$$\mu_k^+(Q) = \int_{Q \cap [f_{\theta_k}' \leq 0]} \mu_k'(t) dt \quad \text{and} \quad \mu_k^-(Q) = - \int_{Q \cap [f_{\theta_k}' > 0]} \mu_k'(t) dt.$$

Recall also that $|\mu_k|(Q) = \mu_k^+(Q) + \mu_k^-(Q)$ and thus

$$\|\mu_k\|_{TV} = |\mu_k|(\mathbb{R}_+) = \mu_k^+(\mathbb{R}_+) + \mu_k^-(\mathbb{R}_+) = \int_{\mathbb{R}_+} t |f_{\theta_k}'(t)| dt.$$

Remark 4.5. It is important to notice that μ_k can still be defined when the density f_{θ_k} is not of class C^1 . More precisely, assume that f_{θ_k} is locally of bounded variation and that the signed measure

$$\mu_k := -t Df_{\theta_k}$$

has finite total variation, where Df_{θ_k} denotes the distributional derivative of f_{θ_k} . Then, for every $\varphi \in C_b(\mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} \varphi(s) d\theta_k(s) = \int_{\mathbb{R}_+} \frac{1}{\tau} \left(\int_0^\tau \varphi(t) dt \right) d\mu_k(\tau).$$

Moreover, μ_k is a signed measure of total mass one.

In what follows, and under the assumptions $(\mathcal{H})$, (11)–(12), we essentially provide two different cases in which (10) holds:

- If $\{\theta_k\}_k$ satisfies a weak LTC condition (namely, for every $T > 0$, $\theta_k([0, T]) \xrightarrow[k \rightarrow \infty]{} 0$) and there exists a uniform value (that is, a long-run time-uniform ε -optimal control, see forthcoming Definition 4.6). This case will be treated in Subsection 4.2.1);
- The sequence $\{\theta_k\}_k$ satisfies a reinforced asymptotic condition, yielding the weak LTC (but not the full LTC), see Subsection 4.2.2.

4.2.1 Existence of a uniform value

We start with the following

Definition 4.6 (uniform value). *We say that our problem admits a long run time-uniform value in $\Omega \subset \mathbb{R}^d$ (or simply, a uniform value in Ω), if for every $\varepsilon > 0$, there exists $T > 0$ such that for every $x_0 \in \Omega$, there exists a control $u \in \mathcal{U}$ such that*

$$V_s(x_0) \leq \gamma_s(x_{x_0,u}) := \frac{1}{s} \int_0^s \ell(x_{x_0,u}(\tau), u(\tau)) d\tau \leq V_s(x_0) + \varepsilon, \quad \text{for all } s \geq T,$$

where $x_{x_0,u}(\cdot)$ is a solution of the differential equation:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), & \text{for a.e. } t \in [0, +\infty) \\ x(0) = x_0 \end{cases}$$

Let us describe a typical situation in which the problem has a uniform value in Ω . (We refer to Remark 4.2 for the definition of invariant set.)

Proposition 4.7 (Ergodic case). *Consider $\Omega \subset \mathbb{R}^d$ an invariant set¹ for (1). Assume that $(\mathcal{H})$ holds on Ω and that $V^* : \Omega \rightarrow \mathbb{R}$ is a constant function (that is, the system is ergodic). Then, the problem admits a uniform value in Ω .*

Proof. Let $\varepsilon > 0$ and consider $T > 0$ such that $\|V_T - V^*\|_\infty < \varepsilon/2$. For each $x_0 \in \Omega$, let $u_0 : [0, T] \rightarrow U$ be an $\varepsilon/2$ -optimal control, so that $|\gamma_T(x_{x_0,u_0}) - V^*| \leq \varepsilon$. Fix $x_0 \in \Omega$. By induction, construct a sequence $\{x_n\}_{n \geq 0} \subset \Omega$ and controls $\{u_n : [0, T] \rightarrow U\}_{n \geq 0}$ such that

$$x_{n+1} = x_{x_n, u_n}(T) \quad \text{and} \quad |\gamma_T(x_{x_n, u_n}) - V^*| \leq \varepsilon.$$

Let $u : [0, +\infty) \rightarrow U$ be the control defined through concatenation of the maps $\{u_n\}_{n \geq 0}$. It follows that $u \in \mathcal{U}$ and for any $k \in \mathbb{N}$

$$|\gamma_{kT}(x_{x_0,u}) - V^*| = \left| \frac{1}{k} (\gamma_T(x_{x_0,u}) + \dots + \gamma_T(x_{x_{k-1},u})) - V^* \right| \leq \frac{1}{k} \sum_{i=0}^{k-1} |\gamma_T(x_{x_i, u_i}) - V^*| \leq \varepsilon.$$

This yields that $V_{kT}(x_0) \leq \gamma_{kT}(x_{x_0,u}) \leq V^*(x_0) + \varepsilon$, for every $k \in \mathbb{N}$. From this inequality and the fact that the control ℓ takes values in $[0, 1]$, we deduce easily that there exists $T' > T$ such that

$$V_t(x_0) \leq \gamma_t(x_{x_0,u}) \leq V^* + 2\varepsilon \leq V_t(x_0) + 3\varepsilon, \quad \text{for all } t > T'.$$

This completes the proof. $\square$

Remark 4.8 (Cases where a uniform value exists). (i). The above proof of Proposition 4.7 holds true in a slightly more general context, namely, when $V^* : \Omega \rightarrow \mathbb{R}$ is invariant for the dynamic $\dot{x} = f(x, u)$. That is, if for every $x_0 \in \Omega$, $u \in \mathcal{U}$ and $t > 0$ we have $V^*(x_0) = V^*(x_{x_0,u}(t))$, then the system admits a uniform value.

(ii). It is worth-noting that a control problem might have a uniform value in Ω even in the case where the limit function V^* is not constant. A typical situation is the case of a system satisfying a nonexpansive-like condition: for continuous lagrangians which do not depend on the control variable, the condition reads:

$$\forall (y_1, y_2) \in \Omega^2, \quad \sup_{u \in U} \inf_{v \in U} \langle y_1 - y_2, f(y_1, u) - f(y_2, v) \rangle \leq 0.$$

¹For instance $\Omega = \mathbb{R}^d$.

We refer the reader to [18] where it is shown that the above condition, as well as several other nonexpansive-like conditions (including cases of control-dependent Lagrangians), ensures the existence of a uniform value.

In what follows, we consider the case $\Omega = \mathbb{R}^d$. As already mentioned in Remark 4.2, our results remain valid when we restrict the functions $\{V_{\theta_k}\}_k$ and V^* to an invariant subset $\Omega \subset \mathbb{R}^d$.

Proposition 4.9. *Assume that the problem has a uniform value and $(\mathcal{H})$ holds. Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be such that (11)–(12) hold and the corresponding sequence $\{\mu_k\}_k$ satisfies:*

$$(14) \quad \lim_{k \rightarrow \infty} |\mu_k|([0, T]) = 0, \quad \text{for every } T > 0.$$

Then

$$(15) \quad \limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \leq 0$$

Proof. Fix $\varepsilon > 0$. Applying Definition 4.6 with $\varepsilon/2$, we obtain $T_0 > 0$ such that for every $x_0 \in \mathbb{R}^d$, there exists a control $u \in \mathcal{U}$ satisfying

$$V_s(x_0) \leq \gamma_s(x_{x_0, u}) \leq V_s(x_0) + \frac{\varepsilon}{2}, \quad \text{for every } s \geq T_0.$$

Since $V_s \rightarrow V^*$ (uniformly as $s \rightarrow +\infty$), by increasing T_0 if necessary, we may choose $T \geq T_0$ such that

$$\|V_s - V^*\|_\infty < \frac{\varepsilon}{2}, \quad \text{for every } s \geq T.$$

Consequently, for every $x_0 \in \mathbb{R}^d$ and corresponding control $u \in \mathcal{U}$ we have:

$$(16) \quad V^*(x_0) - \varepsilon \leq \gamma_s(x_{x_0, u}) \leq V^*(x_0) + \varepsilon, \quad \text{for every } s \geq T.$$

Notice that, for every $k \geq 1$,

$$(17) \quad V_{\theta_k}(x_0) \leq \int_0^{+\infty} \ell(x_{x_0, u}(t), u(t)) d\theta_k(t).$$

Let us set, for simplicity,

$$\ell(t) := \ell(x_{x_0, u}(t), u(t)), \quad \gamma_s := \gamma_s(x_{x_0, u}).$$

We recall that $0 \leq \ell(t) \leq 1$ (whence $0 \leq \gamma_s \leq 1$) and $d\theta_k(t) = f_{\theta_k}(t) dt$. In view of (13) we have:

$$\lim_{t \rightarrow +\infty} t f_{\theta_k}(t) = 0, \quad \text{for every } k \geq 1.$$

Notice that (12) yields:

$$\int_0^{+\infty} t |f'_{\theta_k}(t)| dt < +\infty.$$

Since

$$s\gamma_s = \int_0^s \ell(t) dt,$$

we have $(s\gamma_s)' = \ell(s)$. Integrating by parts in (17), we obtain

$$(18) \quad \begin{aligned} V_{\theta_k}(x_0) &\leq \int_0^{+\infty} \ell(t) f_{\theta_k}(t) dt \\ &= \underbrace{[s\gamma_s f_{\theta_k}(s)]_{s=0}^{s=+\infty}}_{=0} + \int_0^{+\infty} \gamma_s d\mu_k(s) = \int_0^{+\infty} \gamma_s d\mu_k(s). \end{aligned}$$

Since $\mu_k(\mathbb{R}_+) = 1$, we deduce from (18) that

$$V_{\theta_k}(x_0) - V^*(x_0) \leq \int_0^{+\infty} (\gamma_s - V^*(x_0)) d\mu_k(s).$$

On the interval $[0, T]$, recalling that $\gamma_s, V^*(x_0) \in [0, 1]$, we have

$$(19) \quad \int_0^T (\gamma_s - V^*(x_0)) d\mu_k(s) \leq \int_0^T |\gamma_s - V^*(x_0)| d|\mu_k|(s) \leq |\mu_k|([0, T]).$$

On the other hand, it follows from (16) and (12) that

$$(20) \quad \begin{aligned} \int_T^{+\infty} (\gamma_s - V^*(x_0)) d\mu_k(s) &\leq \int_T^{+\infty} |\gamma_s - V^*(x_0)| d|\mu_k|(s) \\ &\leq \varepsilon |\mu_k|([T, +\infty)) \leq \varepsilon \|\mu_k\|_{TV} \leq \varepsilon M. \end{aligned}$$

Combining (19)–(20), we obtain

$$V_{\theta_k}(x_0) - V^*(x_0) \leq |\mu_k|([0, T]) + \varepsilon M.$$

Since T is independent of x_0 , it follows that

$$\sup_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \leq |\mu_k|([0, T]) + \varepsilon M.$$

Taking the limit superior as $k \rightarrow +\infty$ and using Condition (14), we obtain

$$(21) \quad \limsup_{k \rightarrow +\infty} \sup_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \leq \varepsilon M.$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

Remark 4.10. (i) Considering the asymmetric infinite norm (see [8] *e.g.*)

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^d} \max\{0, f(x)\},$$

we observe that (9) and (15) are respectively equivalent to:

$$\limsup_{k \rightarrow \infty} \|V^* - V_{\theta_k}\|_\infty = 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} \|V_{\theta_k} - V^*\|_\infty = 0.$$

Therefore, whenever both conditions hold, we obtain the uniform convergence of the generalized values V_{θ_k} to V^* , that is:

$$\lim_{k \rightarrow \infty} \|V^* - V_{\theta_k}\|_{\infty} = 0.$$

(ii). For $k \geq 1$, let us set $g_k(t) = -t f'_{\theta_k}(t)$, $t \in \mathbb{R}_+$. Then condition (12) (respectively, condition (14)) is equivalent to the fact that for every $T > 0$ we have:

$$\sup_{k \in \mathbb{N}} \|g_k\|_{L^1(\mathbb{R}_+)} < +\infty \quad \text{and respectively} \quad \lim_{k \rightarrow \infty} \|g_k \cdot \mathbb{1}_{[0,T]}\|_{L^1(\mathbb{R}_+)} = 0.$$

In what follows, we denote by $c_k(T)$ (respectively, by c_k) the number of changes in monotonicity of the function f_{θ_k} in $[0, T]$ (respectively, in $\mathbb{R}_+$). Notice that this number can be infinity.

Theorem 4.11 (Majorizing $\limsup V_{\theta_k}$ by V^* – uniform value case). *Suppose that the problem has a uniform value and that $(\mathcal{H})$ holds. Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ satisfy (11)–(12). Then each of the following conditions yields*

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^d} V_{\theta_k}(x) - V^*(x) \leq 0$$

(i). *The functions $\{f_{\theta_k}\}_k$ are nonincreasing and the LTC holds.*

(ii). *For every $T > 0$ it holds:*

$$(22) \quad \lim_{k \rightarrow \infty} \|t f'_{\theta_k}(t) \cdot \mathbb{1}_{[0,T]}(t)\|_{L^1(\mathbb{R}_+)} = 0.$$

(iii). *For every $T > 0$ it holds:*

$$(23) \quad \sup_{k \geq 1} c_k(T) < +\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \|f_{\theta_k} \cdot \mathbb{1}_{[0,T]}\|_{\infty} = 0.$$

Proof. (i). It follows that $f'_{\theta_k}(t) \leq 0$, for all $t \in \mathbb{R}_+$ and $k \geq 1$. Therefore, $\mu_k^-(\mathbb{R}_+) = 0$ and $\mu_k = \mu_k^+ \in \Delta(\mathbb{R}_+)$. We follow the proof of Proposition 4.9 up to formula (21), where we perform a direct calculation:

$$\mu_k^+([0, T]) = \mu_k([0, T]) = \int_0^T -t f'_{\theta_k}(t) dt = \underbrace{[-t f_{\theta_k}(t)]_{t=0}^{t=T}}_{\leq 0} + \int_0^T f_{\theta_k}(t) dt \leq \theta_k([0, T]).$$

The result follows directly from LTC, see (6).

(ii). It is straightforward to see that (22) is equivalent to (14), see also Remark 4.10(ii), therefore the result follows from Proposition 4.9.

(iii). Let us first notice that the second condition in (23) yields that $\theta_k([0, T]) \rightarrow 0$ as $k \rightarrow +\infty$ for all $T > 0$, and that (i) corresponds to the case $c_k = 0$ for all $k \geq 1$. We show that condition (14) holds.

Fix $T > 0$. Since the finite number $c_k(T)$ counts the changes of sign of the continuous function f'_{θ_k} , the interval $[0, T]$ can be written as the union of intervals $[a_i, b_i]$, $i \in \{0, \dots, m_k\}$, with pairwise disjoint interiors, such that f_{θ_k} is monotone on each $[a_i, b_i]$ and $m_k \leq c_k(T)$. Therefore,

$$\begin{aligned} |\mu_k|([0, T]) &= \int_0^T t |f'_{\theta_k}(t)| dt \leq T \sum_{i=0}^{m_k} \int_{a_i}^{b_i} |f'_{\theta_k}(t)| dt \\ &= T \sum_{i=0}^{m_k} |f_{\theta_k}(b_i) - f_{\theta_k}(a_i)| \leq T(m_k + 1) \|f_{\theta_k} \cdot \mathbb{1}_{[0,T]}\|_{\infty}. \end{aligned}$$

Since $\{c_k(T)\}_k$ is bounded above, so is $\{m_k\}_k$. Consequently, the second condition in (23) yields

$$|\mu_k|([0, T]) \xrightarrow[k \rightarrow \infty]{} 0.$$

Thus, condition (14) holds, and the result follows from Proposition 4.9. $\square$

Remark 4.12. The definition 4.6 of the uniform value differs slightly from that used in [18] which is the following :

$$(24) \quad \begin{cases} \forall \varepsilon > 0, \forall x_0, \exists T > 0, \exists u \in \mathcal{U}, \forall s \geq T, \\ V_s(x_0) \leq \gamma_s(x_0, u) := \frac{1}{s} \int_0^s \ell(x_{x_0, u}(\tau), u(\tau)) d\tau \leq V_s(x_0) + \varepsilon; \end{cases}$$

If we suppose that the Lagrangian ℓ is uniformly continuous² with respect to the first variable x , it is straightforward to check that the functions $\{V_T\}_{T>0}$ are continuous and consequently also V^* by our assumption $V_T \rightarrow V^*$ uniformly. When $\bar{\Omega}$ is supposed to be compact and invariant (as it is assumed in [18]) one can easily check that Definition 4.6 and (24) coincide.

We finish this subsection with some results that suggest that the LTC on $\{\theta_k\}_k$ is equivalent to the fact that $V_{\theta_k} \rightarrow V^*$ for any problem that admits a uniform value.

Proposition 4.13. *Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be a sequence of probability measures. Assume that*

- a) *For any $d \geq 1$ and any dynamics and Lagrangian (f, ℓ) satisfying (3) for which the problem admits a uniform value (see Definition 4.6), V_{θ_k} converges uniformly to V^* .*

Then

- b) $\|\theta_k^\sigma\|_{TV} \rightarrow 0$.
c) $\theta_k([0, S]) \rightarrow 0$, for all $S > 0$.
d) For all $S > 0$, $\lim_{k \rightarrow +\infty} \sup_{A \in \mathcal{B}(\mathbb{R}_+), \mathcal{L}(A) \leq S} \theta_k(A) = 0$.

Proof. a) $\implies$ b). Consider $\mathbb{R}$ as ambient space (*i.e.* $d = 1$) and the uncontrolled dynamic on $\mathbb{R}$, $f(x, u) \equiv 1$. For any $k \in \mathbb{N}$, let $N_k \in \mathcal{B}(\mathbb{R}_+)$ be a negligible set such that $\theta_k(N_k) = \|\theta_k^\sigma\|_{TV}$ and set $N := \bigcup_{n \in \mathbb{N}} N_n$. Thus, N is Borel and $\mathcal{L}(N) = 0$. Set $\ell : \mathbb{R} \rightarrow [0, 1]$ be the Borel-function such that $\ell|_N \equiv 1$ and $\ell|_{\mathbb{R} \setminus N} \equiv 0$. It easily follows that $V_T \equiv 0$ and thus uniformly convergent to $V^* \equiv 0$. Moreover, the problem admits (trivially) a uniform value. Also, for $x_0 = 0$ we have that $V_{\theta_k}(x_0) \geq \|\theta_k^\sigma\|_{TV}$, for all $k \in \mathbb{N}$. Thus, since by assumption $\|V_{\theta_k}\|_\infty \rightarrow 0$ as k tends to infinity, we finally deduce that $\|\theta_k^\sigma\|_{TV} \rightarrow 0$.

a) $\implies$ c). Consider $\mathbb{R}$ as ambient space (*i.e.* $d = 1$) and the uncontrolled dynamic on $\mathbb{R}$, $f(x, u) \equiv 1$. Let $S > 0$ and consider the map $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be such that $\ell|_{[0, S]} = 1$ and $\ell|_{\mathbb{R} \setminus [0, S]} = 0$. It easily follows that V_T converges to $V^* \equiv 0$ uniformly and that the problem admits a uniform value. Also, for any $k \in \mathbb{N}$ and $x_0 := 0$, we have that $V_{\theta_k}(x_0) = \theta_k[0, S]$. Since $V_{\theta_k}(x_0) \rightarrow 0$, the conclusion follows.

a) $\implies$ d). Reasoning towards a contradiction, assume that (up to a subsequence) there is $\varepsilon > 0$, $S > 0$ and $\{A_k\}_k \subset \mathcal{B}(\mathbb{R}_+)$ such that $\mathcal{L}(A_k) \leq S$ and $\theta_k(A_k) > \varepsilon$. Thanks to (c) and up to a further subsequence, we can (and shall) assume that

²This is a standard assumption in control theory.

i) $A_k \subset [a_k, a_{k+1}]$ for all $k \in \mathbb{N}$, where

ii) $a_1 = 1$ and $a_{k+1} \geq 10a_k$ for all $k \in \mathbb{N}$.

Let $A := \bigcup_k A_k$ and $\ell : \mathbb{R} \rightarrow [0, 1]$ be the Borel-function defined by $\ell|_A \equiv 1$ and $\ell_{\mathbb{R} \setminus A} = 0$. Take $f(x, u) \equiv 1$. From condition i) and ii) above, it follows that V_T converges uniformly to $V^* \equiv 0$, and the problem admits a uniform control. On the other hand, for $x_0 = 0$, we will have that $V_{\theta_k}(x_0) \geq \varepsilon$ for all $k \in \mathbb{N}$. Thus, a contradiction. $\square$

As noticed in [15, Remark 3.4], the weak LTC condition is equivalent to the LTC for sequences $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ such that each θ_k is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative $f_{\theta_k} := \frac{\partial \theta_k}{\partial \mathcal{L}}$ is nonincreasing. Indeed, for any $Q \in \mathcal{B}(\mathbb{R}_+)$ and any $s > 0$ we have

$$|\theta_k(Q) - \theta_k(Q + s)| = \left| \int_Q f_{\theta_k}(t) - f_{\theta_k}(t + s) dt \right| \leq \int_{\mathbb{R}_+} f_{\theta_k}(t) - f_{\theta_k}(t + s) dt = \theta_k([0, s]).$$

Therefore, we have the following.

Corollary 4.14. *Let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ be a sequence of probability measures such that $\frac{\partial \theta_k^a}{\partial \mathcal{L}}$ is nonincreasing. The following are equivalent.*

- a) *For any $d \geq 1$ and any (f, ℓ) satisfying (3), for which the problem admits a uniform value, V_{θ_k} converges uniformly to V^* .*
- b) *$\{\theta_k\}_k$ satisfies the LTC.*

Proof. b) $\implies$ a). Thanks to Proposition 3.9, we can find $\tilde{\theta}_k$, absolutely continuous with respect to the Lebesgue measure, such that $\|\theta_k - \tilde{\theta}_k\|_{TV} \leq 1/k$ and $f_{\tilde{\theta}_k} := \frac{\partial \tilde{\theta}_k}{\partial \mathcal{L}} \in C^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$. Furthermore, we can (and shall) also assume that $f_{\tilde{\theta}_k}$ is nonincreasing. Thus, again thanks to Proposition 3.9 we have that $\|V_{\theta_k} - V_{\tilde{\theta}_k}\|_{\infty} \rightarrow 0$ and due to Theorem 4.4 and Theorem 4.11, we deduce that $\|V^* - V_{\tilde{\theta}_k}\|_{\infty} \rightarrow 0$. The conclusion follows.

a) $\implies$ b). Thanks to Proposition 4.13, we have that $\|\theta_k^{\sigma}\|_{TV} \rightarrow 0$ and that for any $S > 0$, $\theta_k([0, S]) \rightarrow 0$. Thus, the sequence $\{\rho_k\}_k$, defined by $\rho_k := \frac{1}{\theta_k^a(\mathbb{R}_+)} \theta_k^a$, satisfies LTC (because $\rho_k([0, S]) \rightarrow 0$ for all $S > 0$). Finally, since $\|\theta_k - \rho_k\|_{TV} \rightarrow 0$, we deduce that also the sequence $\{\theta_k\}_{k \geq 1}$ satisfies LTC. $\square$

4.2.2 Reinforced asymptotic condition

In this part, we treat the case where the problem does not have a uniform value. In this case, our analysis will be based on the following result, established in [17, Lemma 5] by M. Oliu-Barton and G. Vigeral. This result will be used as a substitute for the uniform value assumption.

Lemma 4.15. [17] *Assume that $(\mathcal{H})$ holds. Then for every $\varepsilon > 0$ there exists $T_{\varepsilon} > 0$ such that for any $t > T_{\varepsilon}$ and $x_0 \in \mathbb{R}^d$, there exists control $\hat{u} \in \mathcal{U}$ such that*

$$(25) \quad \gamma_s(x_0, \hat{u}) := \frac{1}{s} \int_0^s \ell(x_{x_0, u}(\tau), u(\tau)) d\tau \leq V^*(x_0) + \varepsilon, \quad \text{for all } s \in [\varepsilon t, (1 - \varepsilon)t].$$

Given a sequence of probability measures $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$, we consider the following condition:

(Q) For every $\delta > 0$, there exists $c > 1$ and an increasing sequence $\{s_k\}_k \subset (0, +\infty)$ with $\lim_{k \rightarrow +\infty} s_k = +\infty$ such that

$$(26) \quad \theta_k([s_k, cs_k]) + s_k f_{\theta_k}(s_k) - cs_k f_{\theta_k}(cs_k) \geq 1 - \delta.$$

Proposition 4.16. *Assume that $(\mathcal{H})$ holds. Further, let $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ satisfy (11)–(12) and (Q). If the functions $\{f_{\theta_k}\}_k$ are nonincreasing, then*

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \leq 0.$$

Proof. Notice that, as in the proof of Theorem 4.11(i), since $f'_{\theta_k} \leq 0$, for all $k \geq 1$, the associated measures $\{\mu_k\}_k$ are probability measures. Then, in terms of μ_k condition (Q) reads:

$$(27) \quad \mu_k([s_k, cs_k]) = - \int_{s_k}^{cs_k} \tau f'_{\theta_k}(\tau) d\tau = s_k f_{\theta_k}(s_k) - cs_k f_{\theta_k}(cs_k) + \underbrace{\int_{s_k}^{cs_k} f_{\theta_k}(\tau) d\tau}_{\theta_k([s_k, cs_k])} \geq 1 - \delta.$$

Fix $\delta > 0$ and choose $\varepsilon \in (0, 1)$ sufficiently small to ensure that $(1 - \varepsilon)\varepsilon^{-1} > c$ where $c = c(\delta)$ is given by (Q). Let $T_1 > 0$ be such that $\|V_t - V^*\|_\infty < \varepsilon$ for all $t \geq T_1$. Let $T_\varepsilon \geq T_1$ be given by Lemma 4.15. Let further $k_0 \in \mathbb{N}$ be such that $t_0 := s_{k_0}/\varepsilon > T_\varepsilon$. Then for $k \geq k_0$ we have:

$$t := \frac{s_k}{\varepsilon} > t_0 > T_\varepsilon.$$

Fix $x_0 \in \Omega$ and denote by $\hat{u}$ the control obtained by Lemma 4.15 (depending on x_0 and t) and by $x_{x_0, \hat{u}}(\cdot)$ the associated trajectory starting from x_0 . Observe that $\varepsilon t = s_k$ and $cs_k < (1 - \varepsilon)t$ and consequently $[s_k, cs_k] \subset [\varepsilon t, (1 - \varepsilon)t]$. Then for all $s \in [\varepsilon t, (1 - \varepsilon)t]$ we have:

$$(28) \quad \gamma_s(x_{x_0, \hat{u}}) \leq V^*(x_0) + \varepsilon.$$

To simplify notation, let us denote $\ell(\tau) := \ell(x_{x_0, \hat{u}}(\tau), \hat{u}(\tau))$ and $\gamma_s := \gamma_s(x_{x_0, \hat{u}})$. Recalling that $d\theta_k(\tau) = f_{\theta_k}(\tau) d\tau$ we obtain, arguing as in the proof of Proposition 4.9, that (see (18))

$$(29) \quad V_{\theta_k}(x_0) \leq \int_0^{+\infty} \ell(\tau) f_{\theta_k}(\tau) d\tau = \int_0^\infty \gamma_s d\mu_k(s), \quad \text{for every } k \geq 1.$$

Since μ_k is a probability measure, it follows from (28) that

$$(30) \quad \int_{s_k}^{cs_k} \gamma_s d\mu_k(s) \leq (V^*(x_0) + \varepsilon) \mu_k([s_k, cs_k]) \leq V^*(x_0) + \varepsilon.$$

Recalling that $0 \leq \ell(t) \leq 1$ and consequently, $0 \leq \gamma_s \leq 1$ we deduce that

$$(31) \quad \int_0^{s_k} \gamma_s d\mu_k(s) + \int_{cs_k}^\infty \gamma_s d\mu_k(s) \leq \mu_k([0, s_k] \cup [cs_k, +\infty]) \underbrace{=}_{\mu(\mathbb{R}_+)=1} 1 - \mu_k([s_k, cs_k]) \underbrace{\leq}_{(27)} \delta.$$

Combining (29)–(31) we deduce:

$$V_{\theta_k}(x_0) - V^*(x_0) \leq \varepsilon + \delta$$

and the conclusion follows readily. $\square$

Remark 4.17. The reader might have noticed that the above proof reveals the following fact: if the density functions $\{f_{\theta_k}\}_k$ of the measures $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ are nonincreasing, then condition (Q) is equivalent to the following condition:

(D) For every $\delta > 0$, there exists $c > 1$ and a sequence $\{s_k\}_k \subset (0, +\infty)$ with $\lim_{k \rightarrow \infty} s_k = +\infty$ such that for some $k_0 \in \mathbb{N}$ and all $k \geq k_0$ we have:

$$(32) \quad |\mu_k|(\mathbb{R}_+ \setminus [s_k, cs_k]) \leq \delta.$$

Indeed, (26) of (Q) is readily equivalent to $\mu_k([s_k, cs_k]) \geq 1 - \delta$ and in our case $|\mu_k| = \mu_k \in \Delta(\mathbb{R}_+)$, for every $k \geq k_0$. In general, (D) is more restrictive (it always implies (26)). To see this, assume that (32) holds and notice that:

$$\underbrace{\theta_k([s_k, cs_k]) + s_k f_{\theta_k}(s_k) - cs_k f_{\theta_k}(cs_k)}_{\mu_k([s_k, cs_k])} \geq 1 - \mu_k(\mathbb{R}_+ \setminus [s_k, cs_k]) \geq 1 - |\mu_k|(\mathbb{R}_+ \setminus [s_k, cs_k]) \geq 1 - \delta.$$

Condition (D) allows treating the general case, without requiring any monotonicity on the functions f_{θ_k} . Before we proceed, let us notice that (D) can be directly expressed in terms of the initial data $\{f_{\theta_k}\}_k$ as follows:

(D) For every $\delta > 0$, there exists $c > 1$ and a sequence $\{s_k\}_k \subset (0, +\infty)$ with $\lim_{k \rightarrow \infty} s_k = +\infty$ such that for some $k_0 \in \mathbb{N}$ and all $k \geq k_0$ we have:

$$(33) \quad \lim_{k \rightarrow \infty} s_k = +\infty \quad \text{and} \quad \|t f'_{\theta_k}(t) \cdot \mathbb{1}_{[0, s_k] \cup [cs_k, +\infty]}(t)\|_{L^1(\mathbb{R}_+)} \leq \delta.$$

Theorem 4.18 (Majorizing $\limsup V_{\theta_k}$ by V^* – via condition (D)). *Assume that $(\mathcal{H})$ holds. Suppose that $\{\theta_k\}_k \subset \Delta(\mathbb{R}_+)$ satisfy (11)–(12) and condition (D) holds. Then*

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \leq 0.$$

Proof. Fix $\delta > 0$ and let $c > 1$ and a sequence $\{s_k\}_k$ be given by condition (D). Choose $\varepsilon \in (0, 1)$ sufficiently small so that

$$\frac{1 - \varepsilon}{\varepsilon} > c \quad \text{and} \quad \varepsilon M \leq \delta,$$

where $M > 0$ is given by (12). Let T_ε be given by Lemma 4.15, and choose $T \geq T_\varepsilon$ such that

$$\|V_s - V^*\|_\infty < \varepsilon, \quad \text{for every } s \geq T.$$

Increasing the value of k_0 (given by (33)) if needed, we may assume that $s_k > T$ for every $k \geq k_0$. Fix $k \geq k_0$ and $x_0 \in \mathbb{R}^d$, and set

$$t := \frac{s_k}{\varepsilon} > T > T_\varepsilon.$$

By Lemma 4.15, there exists a control $\hat{u} \in \mathcal{U}$ such that

$$\gamma_s := \gamma_s(x_0, \hat{u}) \leq V^*(x_0) + \varepsilon, \quad s \in [\varepsilon t, (1 - \varepsilon)t].$$

Moreover, since $\varepsilon t = s_k$ and $cs_k < \frac{1-\varepsilon}{\varepsilon} s_k = (1-\varepsilon)t$, we have

$$[s_k, cs_k] \subset [\varepsilon t, (1-\varepsilon)t].$$

Since $s_k > T$ and $V_s(x_0) \leq \gamma_s$, it also follows that

$$V^*(x_0) - \varepsilon \leq V_s(x_0) \leq \gamma_s \leq V^*(x_0) + \varepsilon, \quad \text{for all } s \in [s_k, cs_k].$$

To simplify the notation, set

$$\ell(\tau) := \ell(x_{x_0, \hat{u}}(\tau), \hat{u}(\tau)).$$

Arguing as in the proof of Proposition 4.9 and integrating by parts, we obtain

$$\begin{aligned} V_{\theta_k}(x_0) &\leq \int_0^{+\infty} \ell(\tau) f_{\theta_k}(\tau) d\tau = \int_0^{+\infty} \gamma_s d\mu_k(s) \\ (34) \quad &= \int_0^{s_k} \gamma_s d\mu_k(s) + \int_{s_k}^{cs_k} \gamma_s d\mu_k(s) + \int_{cs_k}^{+\infty} \gamma_s d\mu_k(s). \end{aligned}$$

Since $0 \leq \gamma_s \leq 1$, condition (D) gives

$$\begin{aligned} \int_0^{s_k} \gamma_s d\mu_k(s) + \int_{cs_k}^{+\infty} \gamma_s d\mu_k(s) &= \int_{\mathbb{R}_+ \setminus [s_k, cs_k]} \gamma_s d\mu_k(s) \\ (35) \quad &\leq \int_{\mathbb{R}_+ \setminus [s_k, cs_k]} \gamma_s d|\mu_k|(s) \\ &\leq |\mu_k|(\mathbb{R}_+ \setminus [s_k, cs_k]) \leq \delta. \end{aligned}$$

Using the decomposition $\mu_k = \mu_k^+ - \mu_k^-$ and the preceding bounds on γ_s , we have

$$\begin{aligned} \int_{s_k}^{cs_k} \gamma_s d\mu_k(s) &= \int_{s_k}^{cs_k} \gamma_s d\mu_k^+(s) - \int_{s_k}^{cs_k} \gamma_s d\mu_k^-(s) \\ (36) \quad &\leq (V^*(x_0) + \varepsilon) \mu_k^+([s_k, cs_k]) - (V^*(x_0) - \varepsilon) \mu_k^-([s_k, cs_k]) \\ &= V^*(x_0) \mu_k([s_k, cs_k]) + \varepsilon |\mu_k|([s_k, cs_k]). \end{aligned}$$

Since $\mu_k(\mathbb{R}_+) = 1$, we have $\mu_k([s_k, cs_k]) = 1 - \mu_k(\mathbb{R}_+ \setminus [s_k, cs_k])$. Recalling that $0 \leq V^* \leq 1$, condition (D) yields

$$\begin{aligned} V^*(x_0) \mu_k([s_k, cs_k]) &= V^*(x_0) - V^*(x_0) \mu_k(\mathbb{R}_+ \setminus [s_k, cs_k]) \\ &\leq V^*(x_0) + |\mu_k|(\mathbb{R}_+ \setminus [s_k, cs_k]) \leq V^*(x_0) + \delta. \end{aligned}$$

Furthermore,

$$|\mu_k|([s_k, cs_k]) \leq \|\mu_k\|_{\text{TV}} \leq M.$$

Consequently, (36) gives

$$\int_{s_k}^{cs_k} \gamma_s d\mu_k(s) \leq V^*(x_0) + \delta + \varepsilon M.$$

Combining this inequality with (34)–(35), we obtain

$$V_{\theta_k}(x_0) - V^*(x_0) \leq 2\delta + \varepsilon M \leq 3\delta.$$

The estimate is independent of x_0 . Hence

$$\limsup_{k \rightarrow +\infty} \sup_{x \in \mathbb{R}^d} (V_{\theta_k}(x) - V^*(x)) \leq 3\delta.$$

Since $\delta > 0$ is arbitrary, the conclusion follows. $\square$

Remark 4.19 (Densities of bounded variation). The conclusion of Theorem 4.18 remains valid if the regularity assumption (11) is replaced by the following weaker condition: for every $k \geq 1$, the measure θ_k is absolutely continuous with respect to $\mathcal{L}$, its density f_{θ_k} is locally of bounded variation, and $\mu_k := -t Df_{\theta_k}$ is a finite signed measure satisfying

$$\mu_k(\mathbb{R}_+) = 1 \quad \text{and} \quad \sup_{k \geq 1} \|\mu_k\|_{TV} < +\infty,$$

where Df_{θ_k} denotes the distributional derivative of f_{θ_k} . Indeed, by Remark 4.5, for every bounded measurable function ℓ and

$$\gamma_s := \frac{1}{s} \int_0^s \ell(t) dt,$$

we have

$$\int_{\mathbb{R}_+} \ell(t) d\theta_k(t) = \int_{\mathbb{R}_+} \gamma_s d\mu_k(s).$$

Indeed, the proof of Theorem 4.18 only uses the above representation, the uniform bound on $\|\mu_k\|_{TV}$ and condition (D), therefore it applies without change in this more general setting. In particular, the theorem applies to the piecewise constant and piecewise affine densities considered in forthcoming Sections 4.3.2 and 4.3.5.

4.3 Examples

This section is devoted to examples and counterexamples illustrating our results.

4.3.1 Case $d\theta_k(t) = \lambda_k \rho(\lambda_k t) dt$

Let $\rho \in C^1(\mathbb{R}_+)$ be a nonnegative function satisfying

$$\int_0^\infty \rho(t) dt = 1 \quad \text{and} \quad \|t\rho'(t)\|_{L^1(\mathbb{R}_+)} < +\infty.$$

Let further $\{\lambda_k\}_k \subset (0, +\infty)$ be such that $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$. We will show that the sequence of probability measures $\{\theta_k\}_k$ corresponding to $f_{\theta_k}(t) := \lambda_k \rho(\lambda_k t)$, $k \geq 1$, satisfies (33). It is easy to check that $\{\theta_k\}_k$ satisfies LTC. Therefore, in view of Theorem 4.18 and Theorem 4.4, if the sequence of Cesaro means converges uniformly to some $V^* \in C(\mathbb{R}^d)$ (that is, $\lim_{T \rightarrow \infty} \|V_T - V^*\|_\infty = 0$), then the sequence V_{θ_k} also converges uniformly to V^* . This result has been obtained via a different method in [14].

Indeed, fix $\delta > 0$ and consider $0 < a < b$ such that

$$\int_0^a t|\rho'(t)| dt + \int_b^{+\infty} t|\rho'(t)| dt < \delta.$$

Taking $c := \frac{b}{a}$ and setting $s_k := \frac{a}{\lambda_k}$ we get

$$\begin{aligned} \|t f'_{\theta_k}(t) \cdot \mathbb{1}_{[0, s_k] \cup [cs_k, +\infty]}(t)\|_{L^1(\mathbb{R}_+)} &= \int_0^{s_k} \lambda_k^2 s |\rho'(\lambda_k s)| ds + \int_{cs_k}^{+\infty} \lambda_k^2 s |\rho'(\lambda_k s)| ds \\ &= \int_0^a t |\rho'(t)| dt + \int_b^{+\infty} t |\rho'(t)| dt < \delta \end{aligned}$$

by the change of variable $t = \lambda_k s$. Hence (33) holds true.

Observe that if ρ' is nonpositive, then integrating by parts we obtain

$$\int_0^\infty -t \rho'(t) dt = \|t \rho'(t)\|_{L^1(\mathbb{R}_+)} = 1.$$

The same argument as above shows that condition (Q) holds true.

It is also worth pointing out that an important case is the case of Abel means $\lambda e^{-\lambda t} dt$ which can be viewed as $\lambda \rho(\lambda t)$ with $\rho(t) := e^{-t}$ satisfies the above assumptions. Therefore, our result generalizes the Abelian theorem given in [17].

4.3.2 A decoupled Cesaro-like example

We consider the following sequence of probability measures:

$$d\theta_k(t) = \left(\frac{1}{2a_k} \mathbb{1}_{[0, a_k]}(t) + \frac{1}{2(c_k - b_k)} \mathbb{1}_{[b_k, c_k]}(t) \right) dt,$$

where $0 < a_k < b_k < c_k$, $a_k \rightarrow +\infty$ and $c_k - b_k \rightarrow +\infty$.

One can easily check that the sequence $\{\theta_k\}_k$ satisfies the LTC condition and consequently

$$V^* \leq \liminf_{k \rightarrow \infty} V_{\theta_k} \quad \text{uniformly (cf. Theorem 4.4).}$$

We further assume that

$$(37) \quad \limsup_{k \rightarrow \infty} \frac{c_k}{a_k} < \kappa < +\infty \quad \text{and} \quad \sup_{k \geq 1} \frac{b_k}{c_k - b_k} < +\infty.$$

The second condition guarantees a uniform bound for the total variations of the corresponding sequence of signed measures $\{\mu_k\}_k$, which is required in (12). Indeed, using Remark 4.5, the signed measure μ_k associated with θ_k is given by the formula:

$$\mu_k = \frac{1}{2} \delta_{a_k} - \frac{b_k}{2(c_k - b_k)} \delta_{b_k} + \frac{c_k}{2(c_k - b_k)} \delta_{c_k},$$

where δ_x denotes the Dirac measure at x . Notice in particular that $\mu_k(\mathbb{R}_+) = 1$ and

$$\|\mu_k\|_{\text{TV}} = \frac{1}{2} + \frac{b_k}{2(c_k - b_k)} + \frac{c_k}{2(c_k - b_k)} = 1 + \frac{b_k}{c_k - b_k}.$$

Thus, (12) follows from (37).

Take $c > \kappa$ and set

$$s_k := a_k - \frac{1}{ka_k}.$$

Then $s_k \rightarrow +\infty$. Moreover, for all sufficiently large k ,

$$s_k < a_k < b_k < c_k < cs_k.$$

Consequently,

$$\text{supp}|\mu_k| = \{a_k, b_k, c_k\} \subset [s_k, cs_k],$$

and consequently

$$|\mu_k|(\mathbb{R}_+ \setminus [s_k, cs_k]) = 0$$

for all sufficiently large k . Therefore, condition (D) holds.

Let us finally note that although the density of the measure θ_k is not smooth, the signed measure μ_k is well defined by Remark 4.5 and the proof of Theorem 4.18 applies without change, since it uses only the representation

$$\int_0^{+\infty} \ell(t) d\theta_k(t) = \int_0^{+\infty} \gamma_s d\mu_k(s)$$

and the uniform bound on $\|\mu_k\|_{\text{TV}}$. We thus obtain

$$V^* \geq \limsup_{k \rightarrow \infty} V_{\theta_k} \quad (\text{uniformly}).$$

Combining this inequality with Theorem 4.4 yields the uniform convergence $V_{\theta_k} \rightarrow V^*$.

4.3.3 An example showing that (Q) fails

We provide an example showing that (Q) does not follow from the fact that the sequence $\{\theta_k\}_k$ has the LTC property and nonincreasing densities f_{θ_k} .

Example 4.20. Let $\{T_n\}_n$ be a strictly increasing sequence with $\lim_{n \rightarrow \infty} T_n = \infty$. For $n \geq 1$, we define the nonincreasing density function f_{θ_n} as follows (see Fig. 1):

$$f_{\theta_n}(t) = \begin{cases} \frac{1}{T_n}, & \text{if } 0 \leq t < \frac{T_n}{2}, \\ \frac{1}{nT_n}, & \text{if } \frac{T_n}{2} \leq t < \frac{(n+1)T_n}{2}, \\ 0, & \text{if } t \geq \frac{(n+1)T_n}{2}. \end{cases}$$

In this example, for every $n \geq 1$, the corresponding measure θ_n is a probability measure and that the sequence $\{\theta_n\}_n$ has the LTC property but fails (Q) (and consequently also fails (D)).

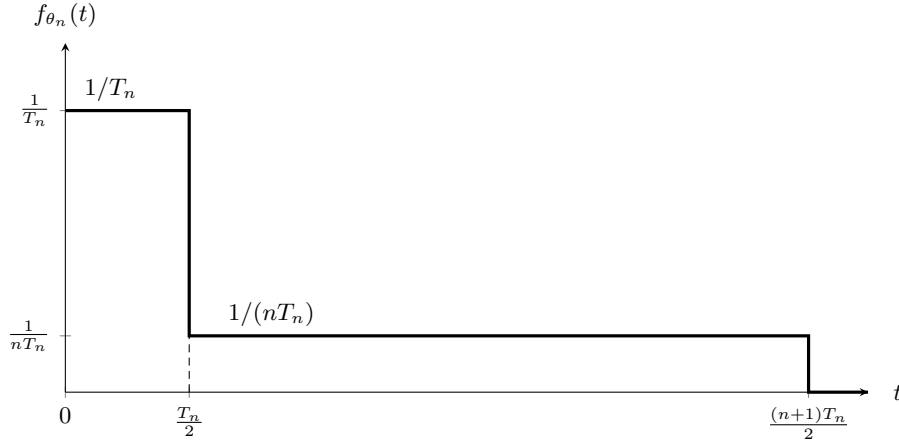


Figure 1: Density function f_{θ_n} (Example 4.20).

Indeed, it is easily seen that the measure associated with θ_n (in the sense of Remark 4.5) is

$$\mu_n = \frac{1}{2}\left(1 - \frac{1}{n}\right) \delta_{\{T_n/2\}} + \frac{1}{2}\left(1 + \frac{1}{n}\right) \delta_{\{(n+1)T_n/2\}}.$$

In particular, μ_n is a probability measure.

Let us show that $\{\theta_n\}_n$ fails property (Q): Let $\delta \in (0, 1/4)$, and assume that there exist $c > 1$ and $\{s_n\}_n \subset (0, +\infty)$, with $s_n \rightarrow +\infty$, such that (26) holds for all sufficiently large n . Equivalently, $\mu_n([s_n, cs_n]) \geq 1 - \delta$, for all sufficiently large n . Fix $n \geq 2$ sufficiently large so that $n + 1 > c$.

Claim. The interval $[s_n, cs_n]$ cannot contain both points $\frac{T_n}{2}$ and $\frac{(n+1)T_n}{2}$.

Indeed, if it contained both points, then $s_n \leq \frac{T_n}{2}$ and $cs_n \geq \frac{(n+1)T_n}{2}$, which would imply $c \geq n + 1$, a contradiction.

Consequently, at least one of the two atoms of μ_n lies outside $[s_n, cs_n]$, and therefore

$$\mu_n(\mathbb{R}_+ \setminus [s_n, cs_n]) \geq \min \left\{ \frac{n-1}{2n}, \frac{n+1}{2n} \right\} = \frac{n-1}{2n} \geq \frac{1}{4} > \delta.$$

It follows that $\mu_n([s_n, cs_n]) < 1 - \delta$, which contradicts property (Q).

Remark 4.21. One can slightly modify the functions $\{f_{\theta_n}\}_n$ in the previous example to make them smooth, without changing the above properties.

4.3.4 An example where (Q) holds but (D) fails

The following example shows that if the monotonicity assumption on the densities $\{f_{\theta_n}\}_n$ is dropped, condition (D) can be strictly stronger than (D).

Example 4.22. Consider the sequence of probability measures $\{\theta_n\}_{n \geq 2}$, with densities:

$$f_{\theta_n}(t) = \frac{1}{n}\left(1 - \frac{1}{n}\right) \mathbb{1}_{[0, n]}(t) + \frac{1}{4n^3} \mathbb{1}_{[n^3, n^3 + 4n^2]}(t), \quad t \in \mathbb{R}_+.$$

Then (12) and (Q) hold, but (D) fails.

The density functions f_{θ_n} are represented in Fig. 2 (the picture is not to scale).



Figure 2: Density function in Example 4.22.

The associated signed measure (in the sense of Remark 4.5) is:

$$\mu_n = -tDf_{\theta_n} = \left(1 - \frac{1}{n}\right) \delta_n - \frac{1}{4} \delta_{\{n^3\}} + \left(\frac{1}{4} + \frac{1}{n}\right) \delta_{\{n^3+4n^2\}}.$$

It follows readily that $\|\mu_n\|_{TV} = \frac{3}{2}$ (hence (12) holds) and $\{\theta_n\}_n$ satisfies LTC. Choosing $s_n = n/2$ and $c > 2$, we verify that (Q) is satisfied. On the other hand, no interval of the form $[s_n, cs_n]$, with $c > 1$ fixed, can simultaneously capture the Dirac mass δ_n and the Dirac masses at n^3 and $n^3 + 4n^2$. It follows easily that condition (D) fails.

4.3.5 A nonmonotone example where conditions (Q) and (D) fail

Let us finally complement the previous examples by illustrating a case where the densities $\{f_{\theta_n}\}_{n \geq 1}$ are non-monotone and both conditions (Q) and (D) fail.

Example 4.23. Let $\{T_n\}_n$ be a strictly increasing sequence with $\lim_{n \rightarrow \infty} T_n = \infty$ and $T_1 > 2$. For $n \geq 1$, set $A_n := 2^{-1}(n+1)T_n$ and define the density f_{θ_n} as follows (see Fig. 3):

$$f_{\theta_n}(t) = \begin{cases} \frac{1}{T_n} & \text{if } 0 \leq t \leq \frac{T_n}{2}, \\ \frac{1}{nT_n} & \text{if } \frac{T_n}{2} \leq t \leq A_n - 1, \\ \frac{2(A_n - t) - 1}{nT_n} & \text{if } A_n - 1 \leq t \leq A_n - \frac{1}{2}, \\ \frac{2(t - A_n) + 1}{nT_n} & \text{if } A_n - \frac{1}{2} \leq t \leq A_n, \\ \frac{A_n + 1 - t}{nT_n} & \text{if } A_n \leq t \leq A_n + 1, \\ 0 & \text{if } t > A_n + 1. \end{cases}$$

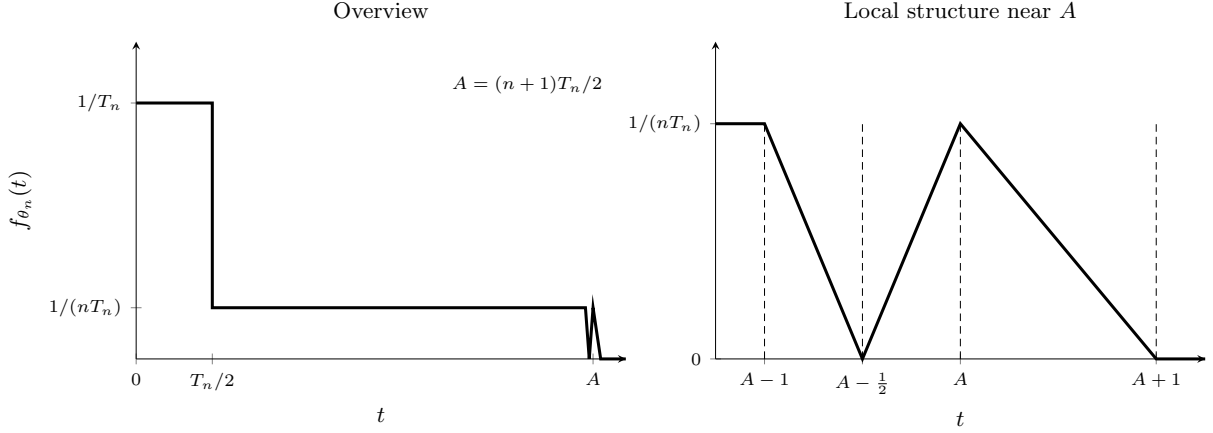


Figure 3: Density function in Example 4.23.

We let the reader verify that $\{\theta_n\}_n \subset \Delta(\mathbb{R}_+)$ and for every $s \in \mathbb{R}_+$,

$$TV_s(\theta_n) \leq \int_0^{+\infty} |f_{\theta_n}(t+s) - f_{\theta_n}(t)| dt \xrightarrow{n \rightarrow +\infty} 0.$$

So, the sequence $\{\theta_k\}_k$ satisfies condition LTC. Moreover, for the signed measure $\mu_n = -t Df_{\theta_n}$ associated to θ_n (in the sense of Remark 4.5) one can easily verify that:

$$\|\mu_n\|_{TV} = 2 + \frac{1}{n} - \frac{1}{2nT_n} < 3, \quad \text{therefore (12) holds.}$$

Finally, let us observe that the jump of f_{θ_n} at $T_n/2$ gives:

$$|\mu_n| \left(\left\{ \frac{T_n}{2} \right\} \right) = \frac{T_n}{2} \left(\frac{1}{T_n} - \frac{1}{nT_n} \right) = \frac{n-1}{2n}.$$

Moreover, since

$$f'_{\theta_n}(t) = -\frac{1}{nT_n}, \quad t \in [A_n, A_n + 1],$$

we have

$$|\mu_n|([A_n, A_n + 1]) = \int_{A_n}^{A_n+1} \frac{t}{nT_n} dt = \frac{A_n + \frac{1}{2}}{nT_n} \geq \frac{1}{2}.$$

Fix $\delta \in (0, 1/4)$ and let $c > 1$. For all sufficiently large n , $A_n > c \frac{T_n}{2}$. Let $s_n > 0$ be arbitrary. If $\frac{T_n}{2} \notin [s_n, cs_n]$, then $|\mu_n|(\mathbb{R}_+ \setminus [s_n, cs_n]) \geq \frac{n-1}{2n} \geq \frac{1}{4} > \delta$. On the other hand, if $\frac{T_n}{2} \in [s_n, cs_n]$, then $s_n \leq T_n/2$ and consequently $cs_n \leq c \frac{T_n}{2} < A_n$. Thus, $[A_n, A_n + 1] \subset \mathbb{R}_+ \setminus [s_n, cs_n]$, and hence

$$|\mu_n|(\mathbb{R}_+ \setminus [s_n, cs_n]) \geq |\mu_n|([A_n, A_n + 1]) \geq \frac{1}{2} > \delta.$$

Therefore, condition (D) fails.

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