Mathematical Programming
https://doi.org/10.1007/s10107-026-02374-w

FULL LENGTH PAPER

Series B

Characterization of regularity via variational stability of alternating projection sequences

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Received: 11 November 2025 / Accepted: 14 May 2026

Abstract

The notion of regular pair (A, B) for two nonempty closed convex subsets A and B of a Hilbert space $\mathcal{H}$ was introduced by Borwein and Bauschke in 1993 to ensure convergence (in norm) of the alternating projection method to some point of the best approximation set. In 2022, De Bernardi and Miglierina showed that regularity of the pair (A, B) guarantees, additionally, the convergence for any variational perturbation of the alternating projection method, provided the corresponding best approximation sets are bounded. In this work, we show that the converse assertion is also true. Moreover, this converse assertion holds without requiring the best approximation sets to be bounded.

Keywords Alternating projection method · Convex feasibility problem · d-stability · regularity · Attouch-Wets convergence

Mathematics Subject Classification Primary: 47J25 · Secondary: 90C25 · 90C48

1 Introduction

Let A and B be two nonempty closed convex sets in a Hilbert space $\mathcal{H}$. The 2-set convex feasibility problem consists of finding a pair of points $\bar{a} \in A$ and $\bar{b} \in B$

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realizing the distance between A and B, that is,

$$\|\bar{a} - \bar{b}\| = \text{dist}(A, B) := \inf_{a \in A} \inf_{b \in B} \|a - b\|.$$

Denoting by $\text{dist}(a, B) := \inf_{b \in B} \|a - b\|$, a necessary and sufficient condition for the existence of such a pair is that the *best approximation sets*

$$E := \{a \in A : \text{dist}(a, B) = \text{dist}(A, B)\}, \quad F := \{b \in B : \text{dist}(b, A) = \text{dist}(A, B)\} \tag{1.1}$$

are nonempty. In this work, we shall assume that this condition is always satisfied and consequently the problem is well-posed. Notice that this happens whenever the intersection of the sets A and B is nonempty (in this case, $E = F = A \cap B$).

The method of alternating projections is the simplest iterative procedure for finding a solution of the convex feasibility problem and it goes back to von Neumann [17]: let us denote by P_A and P_B the projections onto the sets A and B, respectively. Then for any starting point $a_0 \in \mathcal{H}$, consider the *alternating projection sequences* $\{a_n\}_{n \geq 1} \subset A$ and $\{b_n\}_{n \geqslant 1} \subset B$ defined inductively by

$$b_n = P_B(a_{n-1}) \text{ and } a_n = P_A(b_n) \quad (n \in \mathbb{N}). \tag{1.2}$$

If the sequences $\{a_n\}$ and $\{b_n\}$ converge in norm, then we say that the method of alternating projections (also known as the von Neumann method) converges, that is, there exist $\bar{a} \in E$, $\bar{b} \in F$ such that $\lim_{n \to \infty} a_n = \bar{a}$ and $\lim_{n \to \infty} b_n = \bar{b}$. This is always the case in finite dimensions provided the sets E, F in (1.1) are nonempty, while in infinite dimensions, the sequences $\{a_n\}$ and $\{b_n\}$ are only weakly converging, even if $A \cap B \neq \emptyset$, see [5, Theorem 4.8]. Indeed, in the separable Hilbert space ℓ_2, a celebrated example of Hundal in 2004 ([14]) shows that in general the alternating projections method may fail to converge in norm. This example was later simplified by Kopecká [15] and motivated further investigations on conditions that ensure the convergence of the von Neumann method.

Besides the case in which A and B are linear subspaces (this was the original setting studied by von Neumann himself, where obviously one has $0 \in A \cap B = E = F$), one of the best known and most important sufficient conditions for the convergence of such a method is the notion of *regularity* for the pair (A, B) (see forthcoming Definition 2.6), introduced by Bauschke and Borwein in [4]. This notion (which already requires the best approximation sets E and F to be nonempty) guarantees the norm convergence of the sequences defined in (1.2).

In the more recent papers [11–13], the authors studied stability properties of the alternating projection method. More precisely, for any two sequences of nonempty closed convex sets $\{A_n\}_{n \geqslant 1}$ and $\{B_n\}_{n \geqslant 1}$ such that $\lim_{n \to \infty} A_n = A$ and $\lim_{n \to \infty} B_n = B$ for the Attouch-Wets variational convergence (see forthcoming Definition 2.2) and any initial point $a_0 \in \mathcal{H}$, they considered the *perturbed alternating projection sequences* $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, defined as follows:

$$b_n = P_{B_n}(a_{n-1}) \quad \text{and} \quad a_n = P_{A_n}(b_n) \qquad (n \in \mathbb{N}). \tag{1.3}$$

In [11] it was shown that if E, F are nonempty and bounded, then the aforementioned regularity condition on the pair (A, B) not only guarantees the norm convergence of the alternating projection sequence in (1.2), but also the norm convergence of its *variational* version given by (1.3). To be more precise, using the notation of (1.1) let us recall the notion of d-stability introduced in [11].

Definition 1.1 *(d-stability)* We say that the pair (A, B) of two nonempty closed convex subsets of a Hilbert space $\mathcal{H}$ is *d-stable*, whenever all perturbed alternating projection sequences $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ given in (1.3) satisfy

$$\lim_{n\to\infty} \operatorname{dist}(a_n, E) = 0 \quad \text{and} \quad \lim_{n\to\infty} \operatorname{dist}(b_n, F) = 0.$$

The main result of [11] asserts that if the pair (A, B) is regular and the sets E and F are (nonempty and) bounded, then the pair (A, B) is d-stable. The assumption on the boundedness of the best approximation sets E and F is necessary for this statement, since Example 5.2 in [11] provides a regular pair (A, B) with $E = F = A \cap B \neq \emptyset$ being unbounded, which is not d-stable. In the same paper, the authors asked whether the inverse implication holds, stating the following open problem:

Problem 1.2 *Is every d-stable pair (A, B) necessarily regular?*

The main aim of the present paper is to show that Problem 1.2 has a positive answer, and this answer does not require boundedness of the best approximation sets. In particular, we establish the following result.

Theorem A *Let A, B be nonempty closed convex subsets of $\mathcal{H}$. If the pair (A, B) is d-stable, then the pair (A, B) is regular.*

Corollary 1.3 *Let A, B be nonempty closed convex subsets of $\mathcal{H}$ and assume that the best approximation sets E and F are bounded. Then the following are equivalent:*

(i) the pair (A, B) is regular;
(ii) the pair (A, B) is d-stable.

Theorem A establishes implication (ii) $\Longrightarrow$ (i) (see forthcoming Theorems 3.4 and 3.7). The other implication was previously established in [11, Theorem 4.9]. Altogether, Theorem A, Corollary 1.3 and Example 5.2 in [11] provide a complete characterization of regularity by means of variational stability of alternating projection sequences.

Regularity of the pair (A, B) is in fact stronger than mere convergence of alternating projection sequences, see [4, Example 5.5]. Indeed, assuming that the best approximation sets are bounded, regularity turns out to be equivalent to a good behaviour of the alternating projections with respect to perturbations of the two sets A and B, suitable for a study involving perturbations of the original sets, for which it could be easier to compute the projections. In specific cases, regular convex feasibility problems could also lead to the development of algorithmic procedures, designed for the study of convex sets A and B whose description is affected by some errors due to uncertainty in the data. On the other hand, our result suggests avoiding this variational approach in the case of absence of regularity (see Sect. 4 for more details).

We resume this introduction with a brief description of the structure of the paper. In Sect. 2 we fix our notation and review preliminary notions. Definitions and properties of the Hausdorff and, respectively, the Attouch-Wets set convergence are therein recalled, together with relevant properties of metric projections and of alternating projections method in a Hilbert space. In particular, the relationship between regularity and d-stability obtained in [11] is recalled. In Sect. 3, we prove the main result of the paper, namely, the fact that d-stability implies regularity for a given convex feasibility problem. After two key technical lemmas (namely, Lemma 3.1 and Lemma 3.2), the proof of our main result will be divided into two cases: first, we consider the case where $A \cap B$ is the empty set, then we move to the case where $A \cap B$ is nonempty, where the proof becomes more involved. Finally, in Sect. 4 we present some additional remarks, conclusions and open questions.

2 Notation and preliminaries

Throughout this paper, we will always work with a Hilbert space $\mathcal{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. We denote by $\mathbb{B}_{\mathcal{H}}$ and $\mathbb{S}_{\mathcal{H}}$ the closed unit ball and the unit sphere of $\mathcal{H}$, respectively. We denote by $\mathcal{H}^*$ the dual space of $\mathcal{H}$, even if the space $\mathcal{H}^*$ can be represented by $\mathcal{H}$ itself. This notation allows us to emphasize the role of linear functionals in several arguments, making them clearer. Further, we denote by $\mathrm{Fix}(T)$ the set of fixed points of an operator $T : \mathcal{H} \to \mathcal{H}$.

If $\alpha > 0$, $x \in X$, and $A, B \subset \mathcal{H}$, we set as usual

$$x + A := \{x + a;\ a \in A\}, \quad \alpha A := \{\alpha a;\ a \in A\}, \quad A + B := \{a + b;\ a \in A,\ b \in B\}.$$

For any two points $x, y \in \mathcal{H}$, we denote by $[x, y]$ the closed segment in $\mathcal{H}$ with endpoints x and y and we set $(x, y) = [x, y] \setminus \{x, y\}$ for the corresponding "open" segment. The segment $(x, y]$ is defined similarly. For a subset A of $\mathcal{H}$, we denote by $\mathrm{int}\,(A)$ the interior of A. We further define the distance of a point $x \in X$ to a set A as follows:

$$\mathrm{dist}(x, A) := \inf_{a \in A} \|a - x\|.$$

We are going to use the following simplified notation. If f is a real-valued function defined on $\mathcal{H}$, we denote

$$[f \leqslant \alpha] := \{x \in \mathcal{H} :\ f(x) \leqslant \alpha\} \quad \text{for } \alpha \in \mathbb{R}.$$

The sets $[f \geqslant \alpha]$ and $[f = \alpha]$ are defined in a similar way. Finally, we recall that

$$\ker f := \{x \in \mathcal{H} : f(x) = 0\},$$

whenever f is a linear functional defined on $\mathcal{H}$.

2.1 Hausdorff and Attouch-Wets convergence of sequences of sets

Let us now review two notions of convergence for a sequence of sets (for a more detailed overview of this topic, see, e.g., [6]). By $\mathfrak{c}(\mathcal{H})$ we denote the family of all nonempty closed subsets of $\mathcal{H}$. Let us introduce the (extended) Hausdorff metric $\mathrm{D_H}$ in $\mathfrak{c}(\mathcal{H})$. For $A, B \in \mathfrak{c}(\mathcal{H})$, we define the excess of A over B as follows:

$$e(A, B) := \sup_{a\in A} \operatorname{dist}(a, B).$$

Notice that the above definition still makes sense if one of the two sets is empty: indeed, if $A \neq \emptyset$ and $B = \emptyset$ we set $e(A, B) = \infty$, while if $A = \emptyset$ we set $e(A, B) = 0$. Furthermore, we define the Hausdorff distance between sets A and B as follows:

$$\mathrm{D_H}(A, B) := \max\big\{ e(A, B),\ e(B, A) \big\}.$$

The above metric gives rise to our first notion of set convergence.

Definition 2.1 *(Hausdorff convergence)* We say that a sequence $\{A_n\}$ in $\mathfrak{c}(\mathcal{H})$ converges *in the Hausdorff sense* to a set $A \in \mathfrak{c}(\mathcal{H})$ if

$$\lim_{n\to\infty} \mathrm{D_H}(A_n, A) = 0\,.$$

According to [16, Theorem 8.2.12] the sequence $\{A_n\}_{n\geqslant 1}$ in $\mathfrak{c}(\mathcal{H})$ is Hausdorff converging to a set $A \in \mathfrak{c}(\mathcal{H})$ if and only if

$$\sup_{x\in X} |\operatorname{dist}(x, A_n) - \operatorname{dist}(x, A)| \underset{n\to\infty}{\longrightarrow} 0\,.$$

We now consider a weaker notion of convergence, the so-called Attouch-Wets convergence (see, e.g., [16, Definition 8.2.13]). To this end, fix $r > 0$, $A, B \in \mathfrak{c}(\mathcal{H})$ and set:

$$\begin{aligned} e_r(A, B) &:= e(A \cap r\,\mathbb{B}_{\mathcal{H}},\ B)\ \in [0, \infty),\\ \mathrm{D}_{\mathrm{H},r}(A, B) &:= \max\,\{e_r(A, B),\ e_r(B, A)\}. \end{aligned}$$

The above family of pseudo-distances gives rise to the following notion of convergence:

Definition 2.2 *(Attouch-Wets convergence)* We say that a sequence $\{A_n\}_{n\geqslant 1}$ in $\mathfrak{c}(\mathcal{H})$ converges in the *Attouch-Wets sense* to a set $A \in \mathfrak{c}(\mathcal{H})$ if for every $r > 0$ we have:

$$\lim_{n\to\infty} \mathrm{D}_{\mathrm{H},\,r}(A_n, A) = 0\,.$$

The Attouch-Wets convergence can be seen as a "localization" of the Hausdorff convergence. Indeed, according to [16, Theorem 8.2.14], a sequence $\{A_n\}_{n\geqslant 1}$ in $\mathfrak{c}(\mathcal{H})$ is

Attouch-Wets converging to a set $A \in \mathfrak{c}(\mathcal{H})$ if and only if for every $r > 0$

$$\sup_{x \in r\mathbb{B}_{\mathcal{H}}} |\operatorname{dist}(x, A_n) - \operatorname{dist}(x, A)| \underset{n\to\infty}{\longrightarrow} 0\,.$$

The following property relates Hausdorff and Attouch-Wets convergences.

Fact 2.3 *Let $A \in \mathfrak{c}(\mathcal{H})$, $\{A_n\}_{n\geqslant 1} \subset \mathfrak{c}(\mathcal{H})$, $\{r_n\}_n \subset (0, +\infty)$ increasingly converging to $+\infty$, and $\{\delta_n\}_n \subset (0, +\infty)$ converging to 0. Assume that $\mathrm{D_H}(A \cap r_n\mathbb{B}_{\mathcal{H}}, A_n) \leqslant \delta_n$, whenever $n \geqslant 1$. Then, for every $r > 0$, the inequality $\mathrm{D}_{\mathrm{H},r}(A, A_n) \leqslant \delta_n$ eventually holds as $n \to \infty$. In particular, the sequence $\{A_n\}_{n\geqslant 1}$ is Attouch-Wets converging to A.*

Proof Let $r \in \mathbb{N}$. For every $n \in \mathbb{N}$, we clearly have

$$e_r(A_n, A) = e(A_n \cap r\mathbb{B}_{\mathcal{H}}, A) \leqslant e(A_n, A) \leqslant e(A_n, A \cap r_n\mathbb{B}_{\mathcal{H}}) \leqslant \delta_n.$$

Moreover, if $n_0 \in \mathbb{N}$ is such that $r_{n_0} > r$, then for all $n \geqslant n_0$ we have

$$e_r(A, A_n) = e(A \cap r\mathbb{B}_{\mathcal{H}}, A_n) \leqslant e(A \cap r_n\mathbb{B}_{\mathcal{H}}, A_n) \leqslant \delta_n\,.$$

We have shown that $\mathrm{D}_{\mathrm{H},r}(A, A_n) \leqslant \delta_n$ eventually holds as $n \to \infty$. The final conclusion, about the Attouch-Wets convergence of $\{A_n\}_{n\geqslant 1}$, immediately follows. □

2.2 Projections in Hilbert spaces and the alternating projections method

The notions of distance between two convex sets and of projection of a point onto a convex set of a Hilbert space play a fundamental role in our paper. The projection onto a nonempty closed convex subset C maps any point $x_0 \in \mathcal{H}$ to its nearest point in C, denoted by $P_C(x_0)$. We recall the following result, usually named *variational characterization of the projection onto C*. Let $c_0 \in C$ and $x_0 \in \mathcal{H}$. Then $c_0 = P_C(x_0)$ if and only if

$$\langle x_0 - c_0, c - c_0\rangle \leqslant 0, \quad \text{for all } \ c \in C.$$

We recall that the projection P_C is a nonexpansive map from $\mathcal{H}$ to C, i.e., it holds $\|P_C(x) - P_C(y)\| \leqslant \|x - y\|$ (see, e.g., [16, Proposition 10.4.8]).

In the sequel, we shall need the following elementary fact, the proof of which is left to the reader.

Fact 2.4 *Let $C, D \subset \mathcal{H}$ be nonempty closed convex sets such that $C \subset D$. Then:*

(a) *If C is an affine set then P_C is an affine function.*
(b) *If $b, p \in \mathcal{H}$ are such that $p = P_D(b)$ and $p \in C$, then $p = P_C(b)$.*

Given $n \in \mathbb{N}$ and two convex sets $A, B \in \mathfrak{c}(\mathcal{H})$, let us consider the nonexpansive map $\Pi_n^{A,B} : \mathcal{H} \to \mathcal{H}$, defined by

$$\Pi_0^{A,B}(x) = x, \quad \Pi_n^{A,B}(x) = \underbrace{P_A P_B \dots P_A P_B}_{n \text{ times}}(x), \qquad x \in \mathcal{H}.$$

Given A and B in $\mathfrak{c}(\mathcal{H})$, let $\{a_n\}_n$, $\{b_n\}_n$ be the corresponding alternating projection sequences as in (1.2), starting from the initial point $x \in \mathcal{H}$, then we can rewrite them in terms of the above operator $\Pi_n^{A,B}(\cdot)$ as follows:

$$a_n = \Pi_n^{A,B}(x), \qquad b_{n+1} = P_B(\Pi_n^{A,B}(x)).$$

Let us now consider two nonempty closed convex subsets A and B and define the *displacement vector*

$$v := P_{\overline{B-A}}(0) \qquad \text{for the pair } (A, B)\,. \tag{2.1}$$

Recalling (1.1) we have that the (possibly empty) best approximation sets E, F are disjoint if and only if $A \cap B = \emptyset$. Notice that if E and F are nonempty, then there exist $\bar{a} \in E$, $\bar{b} \in F$ such that $v = \bar{b} - \bar{a}$. In case $A \cap B \neq \emptyset$, then $E = F = A \cap B$ and the displacement vector for the pair (A, B) is null. Furthermore:

Fact 2.5 ([4, Fact 1.1]) *Suppose that $\mathcal{H}$ is a Hilbert space and that A, B are nonempty closed convex subsets of $\mathcal{H}$, such that the corresponding best approximation sets E and F are nonempty. Then:*

(i) $\|v\| = \operatorname{dist}(A, B)$ *and* $E + v = F$;
(ii) $E = \operatorname{Fix}(P_A P_B) = A \cap (B - v)$ *and* $F = \operatorname{Fix}(P_B P_A) = B \cap (A + v)$;
(iii) For every $\bar{a} \in E$ *and* $\bar{b} \in F$ *we have:*

$$P_B\bar{a} = P_F\bar{a} = \bar{a} + v \qquad \textit{and} \qquad P_A\bar{b} = P_E\bar{b} = \bar{b} - v.$$

2.3 Regularity and *d*-stability for a pair of convex sets

We shall now state the notion of regularity for a pair of nonempty closed convex sets A and B, which has been introduced in [4] to ensure convergence in norm for the alternating projection algorithm (see also [7]).

Definition 2.6 *(regular pair)* Let A and B be two nonempty closed convex subsets of a Hilbert space $\mathcal{H}$, such that the corresponding best approximation sets E and F are nonempty. The pair (A, B) is called *regular* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in \mathcal{H}$ we have:

$$\max\{\operatorname{dist}(x, A), \operatorname{dist}(x, B - v)\} \leqslant \delta \quad \Longrightarrow \quad \operatorname{dist}(x, E) \leqslant \varepsilon\,. \tag{2.2}$$

We refer the reader to [4] for concrete examples of pair of sets satisfying (or failing to satisfy) regularity property.

The next result follows from [4, Theorem 3.7] and [11, Theorem 4.9]. Indeed, recalling Definition 1.1 we have:

Theorem 2.7 *Let A, B be nonempty closed convex subsets of a Hilbert space $\mathcal{H}$ so that the pair (A, B) is regular. The following assertions hold:*

(i) the alternating projection method converges in norm;
(ii) if E, F are bounded, the pair (A, B) is d-stable.

3 Main results

In this section, our aim is to prove Theorem A. To this end, we shall need some intermediate technical lemmas, which we are going to obtain progressively. Let us start with the following important intermediate result that provides a framework under which a sequence generated by the alternating projection method remains in a 2-dimensional space.

Lemma 3.1 (keeping iterations in a 2-dim space) *Let $\gamma \geqslant 0$, let z, w be two nonzero elements of a Hilbert space $\mathcal{H}$ such that $\langle z, w\rangle = 0$ and define:*

$$u := \tfrac{z}{\|z\|}, \quad \alpha := \tfrac{\|z\|}{\|w\|}, \quad x \mapsto f(x) = \langle u, x\rangle \quad , \quad x \mapsto \widehat{f}(x) = \langle u + \alpha \tfrac{w}{\|w\|}, x\rangle. \quad (3.1)$$

Let further $\widehat{A}, \widehat{B}$ be nonempty closed convex subsets of $\mathcal{H}$ such that

$$[z, w] \subseteq \widehat{A} \subseteq \left[\widehat{f} \geqslant \widehat{f}(w)\right] \quad \textit{and} \quad [-\gamma u, w - \gamma u] \subseteq \widehat{B} \subseteq [f \leqslant -\gamma]. \quad (3.2)$$

Then starting from $a_0 = 0$ the alternating projection method generates sequences

$$a_n = \Pi_n^{\widehat{A},\widehat{B}}(0), \quad b_n = P_{\widehat{B}}(a_{n-1}), \quad n \geqslant 1,$$

such that if $\gamma = 0$ we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = w \in \widehat{A} \cap \widehat{B},$$

while if $\gamma \neq 0$, there exists $m \in \mathbb{N} \cup \{0\}$ satisfying

$$\|a_m - w\| \leqslant \gamma\, \alpha \, \frac{\|w - z\|}{\|w\|}.$$

Proof Let us denote by $\mathbf{h}$ the half-line with origin $w - \gamma u$ and passing through the point $-\gamma u$, and by $\mathbf{t}$ the line passing through z and w. Then denote by b^* the unique point on $\mathbf{h}$ such that $P_{\mathbf{t}}(b^*) = w$. Let us also denote by a^* the unique point on $\mathbf{t}$ such that $b^* = P_{\mathbf{h}}(a^*)$. If we define $\theta = \arctan \alpha$, since

$$\frac{\gamma}{\|b^* - w\|} = \cos\theta = \frac{1}{\sqrt{1+\alpha^2}} = \frac{\|w\|}{\sqrt{\|w\|^2 + \|z\|^2}} = \frac{\|w\|}{\|w - z\|},$$

we obtain

$$\|a^* - w\| = \underbrace{\left(\frac{\|z\|}{\|w\|}\right)}_{\alpha := \tan\theta} \|b^* - w\| = \alpha\, \gamma \, \frac{\|w - z\|}{\|w\|}.$$

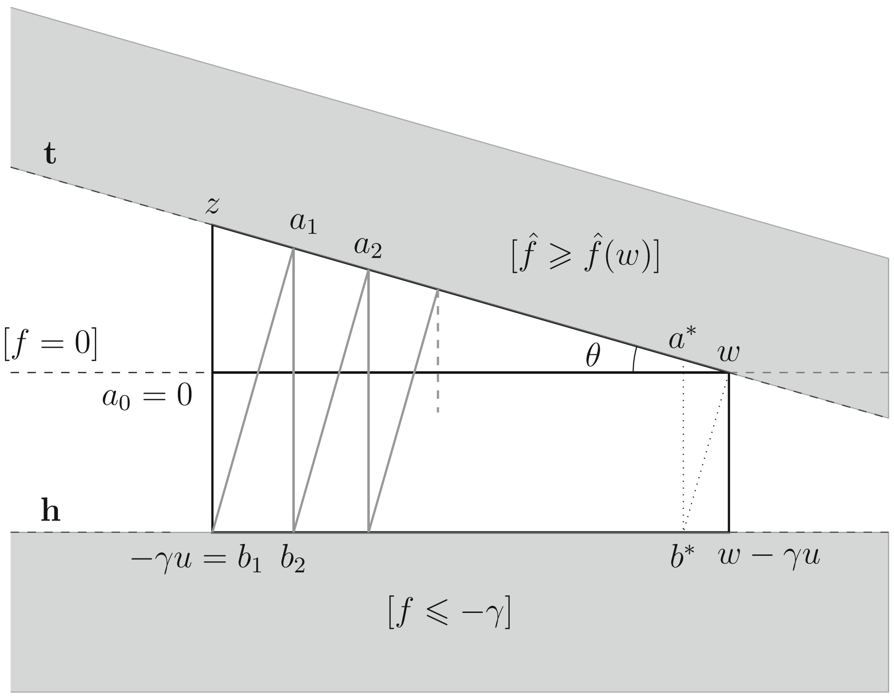

Fig. 1 Proof of Lemma 3.1

Note that if $b^* \notin [-\gamma u, w - \gamma u]$ then we necessarily have $z \in [a^*, w]$ and hence

$$\|a_0 - w\| = \|w\| \leqslant \|z - w\| \leqslant \|a^* - w\| = \gamma\, \alpha\, \frac{\|w - z\|}{\|w\|},$$

and our conclusion holds with $m = 0$. Hence, we can suppose without loss of generality that $b^* \in [-\gamma u, w - \gamma u]$.

Observe that

$$P_{[f \leqslant -\gamma]}(z) = P_{[f = -\gamma]}(z) = -\gamma u \quad \text{and} \quad P_{[f \leqslant -\gamma]}(w) = P_{[f = -\gamma]}(w) = w - \gamma u.$$

Since the projection to the affine subspace $[f = -\gamma]$ is an affine function (Fact 2.4(a)), for every $x \in [z, w]$, we have $P_{[f \leqslant -\gamma]}(x) \in [-\gamma u, w - \gamma u]$. Therefore, we infer from Fact 2.4(b) that, for every $x \in [z, w]$, $P_{\widehat{B}}(x) \in [-\gamma u, w - \gamma u]$. A similar argument shows that, for every $x \in [-\gamma u, b^*]$, we have $P_{\widehat{A}}(x) \in [z, w]$.

We now consider two cases:

(i) If $\gamma = 0$, then $w = b^* \in \widehat{A} \cap \widehat{B}$. Then, setting $\widetilde{A} := [z, w]$ and $\widetilde{B} := [0, w]$ we deduce that $a_n := \Pi_n^{\widehat{A},\widehat{B}}(a_0) = \Pi_n^{\widetilde{A},\widetilde{B}}(a_0)$. Since $\Pi_n^{\widetilde{A},\widetilde{B}}(a_0) \to w$ as $n \to \infty$ (see, e.g., [4, Fact 1.2]), the assertion follows.

(ii) If $\gamma \neq 0$ then $b^* \neq w - \gamma u$, that is, we are in the situation described in Fig. 1. Notice that, as long as the elements $b_n := P_{\widehat{B}}(a_{n-1})$ are not in the segment

$(b^*, w - \gamma u]$, their projections $a_n := P_{\widehat{A}}(b_n)$ are in the segment $[z, w]$, and these projections lie in the two-dimensional subspace determined by the points $0, z, w$. Now, let $m \in \mathbb{N} \cup \{0\}$ be the first non-negative integer such that $b_{m+1} := P_{\widehat{B}}(a_m) \in [b^*, w - \gamma u]$. Then we easily get

$$\|a_m - w\| \leqslant \|a^* - w\| = \gamma\, \alpha\, \frac{\|w - z\|}{\|w\|},$$

the proof is complete. □

Remark. Fig. 1 and its role in the proof of Lemma 3.1 were inspired by [15, Fig. 3].

We shall now consider two nonempty closed convex sets A and B that are separated by a functional and show that, provided one of them is bounded, we can perturb the sets to obtain a setting in which the previous lemma can be applied (Fig. 2).

Lemma 3.2 (Steering the iterations) *Let A, B be closed nonempty convex subsets of $\mathcal{H}$ such that for some $f \in \mathbb{S}_{\mathcal{H}^*}$, $\beta \in \mathbb{R}$, $\gamma \geqslant 0$, and $r \geqslant 1$ we have:*

$$\sup f(B) \leqslant \beta - \gamma \leqslant \beta \leqslant \inf f(A) \qquad \textit{and} \qquad A \subset r\mathbb{B}_{\mathcal{H}}\,.$$

Let further $p, w \in [f = \beta]$, $p \neq w$ and suppose that for some $\delta > 0$

$$(p + \delta\mathbb{B}_{\mathcal{H}}) \cap A \neq \emptyset, \quad (w + \delta\mathbb{B}_{\mathcal{H}}) \cap A \neq \emptyset, \quad (p - \gamma u + \delta\mathbb{B}_{\mathcal{H}}) \cap B \neq \emptyset,$$
$$(w - \gamma u + \delta\mathbb{B}_{\mathcal{H}}) \cap B \neq \emptyset,$$

where $u \in \mathbb{S}_{\mathcal{H}}$ represents the functional f.

Then there exist $m \in \mathbb{N} \cup \{0\}$ and closed convex sets $\widehat{A}, \widehat{B}$ such that

$$\mathrm{D_H}(A, \widehat{A}) \leqslant 3\delta\,, \quad \mathrm{D_H}(B, \widehat{B}) \leqslant 3\delta$$

and such that, for $a_0 := p$, $a_n := \Pi_n^{\widehat{A},\widehat{B}}(a_0)$ and $b_n := P_{\widehat{B}}(a_{n-1})$ ($n \in \mathbb{N}$), we have

(i) the finite sequence $\{b_n\}_{n=1}^{m+1}$ is contained in $[f = \beta - \gamma]$;

(ii) $a_m := \Pi_m^{\widehat{A},\widehat{B}}(a_0) \in (w + \delta\mathbb{B}_{\mathcal{H}}) \cap \widehat{A}$.

Proof Without loss of generality, we may assume that $p = 0$ and consequently $\beta = 0$, $\langle u, w\rangle = 0$, $w \in \ker f$ and $A \subset [f \geqslant 0]$. We set:

$$\widehat{B} = (B + 3\delta\mathbb{B}_{\mathcal{H}}) \cap \big[f \leqslant -\gamma\big].$$

Since $\mathrm{D_H}(B, B + 3\delta\mathbb{B}_{\mathcal{H}}) \leqslant 3\delta$ and $B \subseteq [f \leqslant -\gamma]$, we have $B \subset \widehat{B} \subset B + 3\delta\mathbb{B}_{\mathcal{H}}$ and therefore

$$\mathrm{D_H}(B, \widehat{B}) \leqslant 3\delta.$$

Set a positive number ε_0 so that

$$\varepsilon_0 \leqslant \min\{1, \|w\|\} \qquad \text{(the exact value of } \varepsilon_0 \text{ will be fixed later).}$$

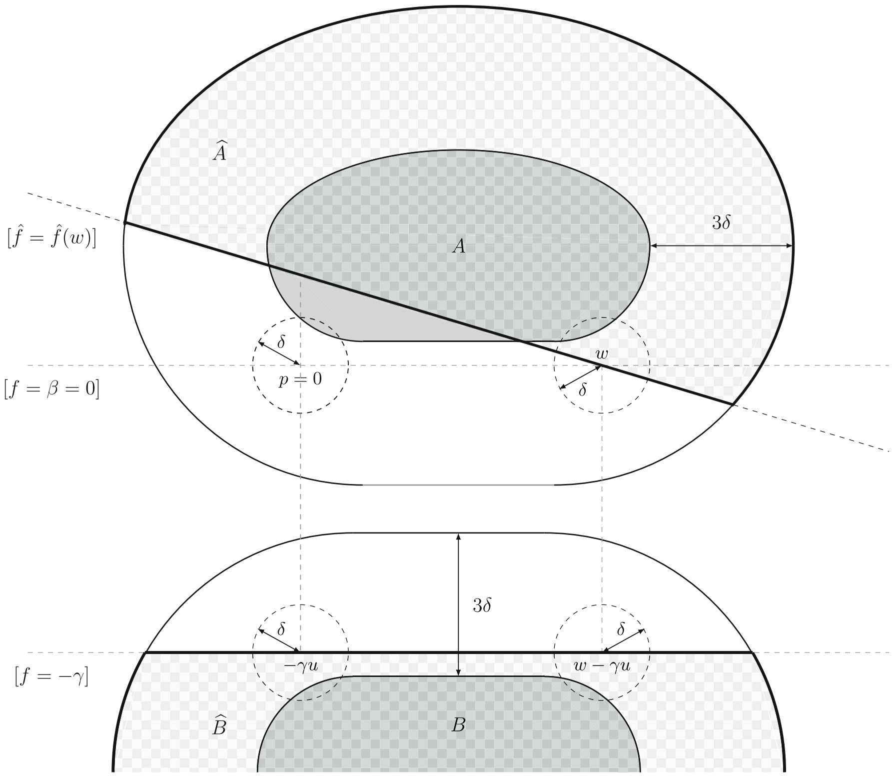

Fig. 2 Construction of $\widehat{A}$ and $\widehat{B}$

Let $u \in \mathbb{S}_{\mathcal{H}}$ representing the functional $f \in \mathbb{S}_{\mathcal{H}^*}$ (that is, $f(\cdot) = \langle u, \cdot\rangle$) and define:

$$\widehat{f}(x) = \langle u + \alpha \frac{w}{\|w\|}, x\rangle = f(x) + \alpha\langle \frac{w}{\|w\|}, x\rangle, \quad x \in \mathcal{H}, \qquad \text{where} \quad \alpha := \frac{\delta\varepsilon_0}{r\|w\|}. \tag{3.3}$$

Since $\|u\| = 1$ and $\langle u, w\rangle = 0$, we have $\|\widehat{f}\| > 1$. Moreover, for every $a \in A$ we have:

$$\widehat{f}(a) = f(a) + \alpha\langle \frac{w}{\|w\|}, a\rangle \geqslant -\alpha\|a\|. \tag{3.4}$$

We now consider the set

$$\widehat{A} = (A + 3\delta\mathbb{B}_{\mathcal{H}}) \bigcap \left[\widehat{f} \geqslant \widehat{f}(w)\right].$$

Claim. $\mathrm{D_H}(A, \widehat{A}) \leqslant 3\delta$.

Proof of the claim. Since $\widehat{A} \subset A + 3\delta\mathbb{B}_{\mathcal{H}}$, we have $e(\widehat{A}, A) \leqslant 3\delta$. Let now $a \in A$. If $a \in \left[\widehat{f} \geqslant \widehat{f}(w)\right]$, then $a \in \widehat{A}$, while if $\widehat{f}(a) < \widehat{f}(w) = \frac{\delta\,\varepsilon_0}{r} = \alpha\|w\|$, we set

$$\widetilde{a} = P_{[\widehat{f} \geqslant \widehat{f}(w)]}(a) = P_{[\widehat{f} = \widehat{f}(w)]}(a).$$

Then $\widehat{f}(\widetilde{a}) = \widehat{f}(w) = \delta\varepsilon_0 r^{-1}$. Recalling that $\|a\| \leqslant r$, for all $a \in A$ and $\varepsilon_0 \leqslant 1 \leqslant r$, we deduce from (3.4) that

$$\|\widetilde{a} - a\| = \frac{\left|\widehat{f}(w) - \widehat{f}(a)\right|}{\|\widehat{f}\|} < \widehat{f}(w) - \widehat{f}(a) \leqslant \delta \left(\frac{\varepsilon_0}{r}\right) + \delta\, \varepsilon_0 \left(\frac{\|a\|}{r}\right) \leqslant 2\,\delta.$$

It follows that $\widetilde{a} \in \widehat{A}$, whence $e(A, \widehat{A}) \leqslant 2\delta$ and the claim follows.

We finally set

$$z := \left(\frac{\delta\varepsilon_0}{r}\right) u = \alpha \|w\|\, u\,. \tag{3.5}$$

By the definition of the sets $\widehat{A}$ and $\widehat{B}$, we deduce that

$$[z, w] \subset \widehat{A} \qquad \text{and} \qquad [-\gamma u, w - \gamma u] \subset \widehat{B}.$$

Therefore, we can apply Lemma 3.1 for the sets $\widehat{A}$, $\widehat{B}$ (note that α is indeed equal to $\frac{\|z\|}{\|w\|}$) to conclude that there exists $m \in \mathbb{N}$ satisfying

$$\| \underbrace{\Pi_m^{\widehat{A},\widehat{B}}(a_0)}_{a_m} - w\| \leqslant \gamma\, \frac{\|z\|}{\|w\|}\, \frac{\|w - z\|}{\|w\|} \leqslant \left(\gamma\, \frac{\|w\| + \|z\|}{\|w\|^2}\, \frac{\varepsilon_0}{r}\right)\delta\,.$$

Shrinking the value of ε_0 and recalling (3.5) we ensure that the above quantity is less or equal to δ. The proof is complete. □

We now establish our main result under the additional assumption that the sets A and B are disjoint (and the best approximation sets E and F are nonempty). This assumption considerably simplifies the proof, allowing us to outline the geometrical features of the argument.

Given a pair (A, B) of nonempty closed convex sets in $\mathcal{H}$, we recall from (1.1) the definition of the best approximation sets E, F and from (2.1) the definition of the displacement vector $v := P_{\overline{B-A}}(0)$. We start with the following technical lemma.

Lemma 3.3 (lack of regularity) *Let A, B be two closed convex sets in $\mathcal{H}$. Assume that E, F are nonempty and disjoint and that the pair (A, B) is not regular.*

Then there exist $\varepsilon_0 > 0$, a sequence $\{\delta_n\}_{n\geqslant 1} \subset (0, +\infty)$ with $\delta_n \to 0$, and a sequence $\{w_n\}_{n\geqslant 1} \subset \mathcal{H}$ such that for every $n \in \mathbb{N}$ we have:

- *(i)* $\varepsilon_0 < \operatorname{dist}(w_n, E) \leqslant \varepsilon_0 + 1$;
- *(ii)* $\max\Big\{\operatorname{dist}(w_n, A),\ \operatorname{dist}(w_n, B - v)\Big\} \leqslant \delta_n$;
- *(iii)* $\langle v, w_n\rangle = \langle v, \bar{a}\rangle$ *for some (equivalently, for all) $\bar{a} \in E$.*

Proof Since the pair (A, B) is not regular, there exist $\{w'_n\}_{n\geqslant 1} \subset \mathcal{H}$ and $\varepsilon_0 > 0$ such that

$$\operatorname{dist}(w'_n, E) > 3\varepsilon_0, \quad \text{for all } n \geqslant 1 \quad \text{and} \quad \lim_{n\to\infty} \max\left\{\operatorname{dist}(w'_n, A),\ \operatorname{dist}(w'_n, B - v)\right\} = 0.$$

Since E, F are nonempty and $F = E + v$, we can separate A and $B - v$ with an affine hyperplane H orthogonal to the displacement vector v, containing the set E. Then setting $w_n'' := P_H(w_n')$, one has

$$\|w_n' - w_n''\| \leqslant \max\{\mathrm{dist}(w_n', A), \mathrm{dist}(w_n', B - v)\},$$

yielding

$$\lim_{n\to\infty} \|w_n' - w_n''\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \max\left\{\mathrm{dist}(w_n'', A),\ \mathrm{dist}(w_n'', B - v)\right\} = 0.$$

Then for some $n_0 \in \mathbb{N}$ sufficiently large and all $n \geq n_0$ we have:

$$\mathrm{dist}(w_n'', E) \geqslant \mathrm{dist}(w_n', E) - \|w_n' - w_n''\| > 2\varepsilon_0\,.$$

We now pick $w_n \in [w_n'', P_E(w_n'')]$ so that $\varepsilon_0 < \mathrm{dist}(w_n, E) < \varepsilon_0 + 1$. Finally, by convexity of the sets A and B we have

$$\delta_n := \max\{\mathrm{dist}(w_n, A), \mathrm{dist}(w_n, B - v)\} \leqslant \max\left\{\mathrm{dist}(w_n'', A),\ \mathrm{dist}(w_n'', B - v)\right\}$$

and the proof is complete. $\square$

We are now ready to establish the result in the special case of the aforementioned assumption that the best approximation sets are (nonempty and) disjoint.

Theorem 3.4 (main result: case $A \cap B = \emptyset$) *Let A, B be nonempty closed convex subsets of $\mathcal{H}$ such that E, F are nonempty and disjoint. If the pair (A, B) is not regular, then (A, B) is not d-stable.*

Proof Notice that $v \neq 0$ (displacement vector) and that we can assume that for any initial point x, the alternating projection sequence with respect to the sets A, B satisfies

$$\lim_{n\to\infty} \mathrm{dist}\left(\Pi_n^{A,B}(x)\,,\ E\right) = 0, \tag{3.6}$$

because, otherwise, the pair (A, B) will already be not d-stable. Moreover, there is no loss of generality to assume $0 \in E$. Let $f \in \mathcal{H}^*$ be the linear functional defined by $f(\cdot) := \langle \cdot, -v\rangle$.

By Lemma 3.3 (lack of regularity), there exist $\varepsilon_0 > 0$, a sequence of positive numbers $\{\delta_n\}_n$ with $\delta_n \to 0$ and a sequence $\{w_n\}_{n\geqslant 1} \subset \ker f$ such that

$$\mathrm{dist}(w_n, E) > 3\varepsilon_0 \qquad \text{and} \qquad \max\left\{\mathrm{dist}(w_n, A),\ \mathrm{dist}(w_n, B - v)\right\} \leqslant \delta_n \underset{n\to\infty}{\longrightarrow} 0\,.$$

We can assume that $\delta_n \leqslant \varepsilon_0$, for all $n \in \mathbb{N}$.

Take an arbitrary $q_0 \in \mathcal{H}$: in view of (3.6), there exist $n_1 \in \mathbb{N}$ and $p_1 \in E$, both depending on ε_0, such that

$$\widetilde{p}_1 := \Pi_{n_1}^{A,B}(q_0) \in (p_1 + \varepsilon_0 \mathbb{B}_{\mathcal{H}}) \cap A\,. \tag{3.7}$$

We shall now show that for any $r_1 > 0$ sufficiently large, there exist two closed convex sets $\widehat{A}_1$ and $\widehat{B}_1$ which are $3\delta_1$-close (with respect to the Hausdorff distance $\mathrm{D_H}$) to the sets $A \cap r_1\mathbb{B}_{\mathcal{H}}$ and B, respectively, such that the iterations of the corresponding alternating projection method, starting from $\widetilde{p}_1$, bring us to some point q_1 sufficiently close to the point w_1.

To this end, we first show that starting from the point $p_1 \in (\widetilde{p}_1 + \varepsilon_0\mathbb{B}_{\mathcal{H}}) \cap \ker f$ we can get sufficiently close to the point w_1: indeed, notice that $p_1 \in E \subset \ker f$ and $w_1 \in \ker f$. Moreover, the set $w_1 + \delta_1\mathbb{B}_{\mathcal{H}}$ has nontrivial intersection with both sets A and $B - v$. Let $\widetilde{a}_1 \in A$ be such that $\|\widetilde{a}_1 - w_1\| \leqslant \delta_1$, and pick $r_1 \geqslant \max\{1, \|\widetilde{a}_1\|, \|p_1\|\}$.

Therefore, we can apply Lemma 3.2 to the sets $A \cap r_1\mathbb{B}_{\mathcal{H}}$ and B, for

$$\delta{:=}\delta_1 \leqslant \varepsilon_0, \quad r{:=}r_1, \quad \gamma{:=}\|v\|, \quad \beta{:=}0$$

to obtain $m_1 \in \mathbb{N} \cup \{0\}$ and closed convex sets $\widehat{A}_1$ and $\widehat{B}_1$ such that

$$\mathrm{D_H}(A \cap r_1\mathbb{B}_{\mathcal{H}}, \widehat{A}_1) \leqslant 3\delta_1, \qquad \mathrm{D_H}(B, \widehat{B}_1) \leqslant 3\delta_1$$

and

$$\widetilde{w}_1{:=}\Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(p_1) \in w_1 + \delta_1\mathbb{B}_{\mathcal{H}} \subset w_1 + \varepsilon_0\mathbb{B}_{\mathcal{H}}.$$

Starting now the iterations from the point $\widetilde{p}_1$ satisfying (3.7), since $\|p_1 - \widetilde{p}_1\| \leqslant \varepsilon_0$ and the projection operators $P_{\widehat{A}_1}$, $P_{\widehat{B}_1}$ are non-expansive, we deduce

$$q_1{:=}\Pi_{m_1}^{\widehat{A}_1,\widehat{B}_1}(\widetilde{p}_1) \in \widetilde{w}_1 + \varepsilon_0\mathbb{B}_{\mathcal{H}}.$$

Consequently, $\|q_1 - w_1\| \leqslant 2\varepsilon_0$. Notice that since $\operatorname{dist}(w_1, E) > 3\varepsilon_0$ we get $\operatorname{dist}(q_1, E) > \varepsilon_0$.

Concatenating the sequences:

$$\{\Pi_n^{A,B}(q_0)\}_{n\leqslant n_1} \quad (\text{joining } q_0 \text{ to } \widetilde{p}_1) \quad \text{and} \quad \{\Pi_n^{\widehat{A}_1,\widehat{B}_1}(\widetilde{p}_1)\}_{n\leqslant m_1} \quad (\text{joining } \widetilde{p}_1 \text{ to } q_1)$$

we obtain a finite sequence of variational alternating projections whose last point q_1 is outside the set $E + \varepsilon_0\mathbb{B}_{\mathcal{H}}$.

Iterating the same argument as above, we can inductively obtain a sequence of sets $\widehat{A}_h, \widehat{B}_h \subset \mathcal{H}$, together with points $q_h, \widetilde{p}_h \in \mathcal{H}$, $p_h \in E$, $\widetilde{a}_h \in A$, positive numbers r_h, and non-negative integers n_h, m_h such that, for every $h \in \mathbb{N}$, we have:

(i) $\widetilde{p}_h = \Pi_{n_h}^{A,B}(q_{h-1})$ and $\|p_h - \widetilde{p}_h\| < \varepsilon_0$;

(ii) $\|\widetilde{a}_h - w_h\| \leqslant \delta_h$ and $\max\{\|p_h\|, \|\widetilde{a}_h\|, h\} \leqslant r_h$,

(iii) $\mathrm{D_H}\left(A \cap r_h\mathbb{B}_{\mathcal{H}}, \widehat{A}_h\right) \leqslant 3\delta_h$ and $\mathrm{D_H}\left(B, \widehat{B}_h\right) \leqslant 3\delta_h$;

(iv) $q_h = \Pi_{m_h}^{\widehat{A}_h,\widehat{B}_h}(\widetilde{p}_h) = \Pi_{m_h}^{\widehat{A}_h,\widehat{B}_h}\left(\Pi_{n_h}^{A,B}(q_{h-1})\right)$ and $\operatorname{dist}(q_h, E) \geqslant \varepsilon_0$.

To prove that the pair (A, B) is not d-stable, let us consider the sequence $\{A_k\}_k$ of closed convex sets in $\mathcal{H}$ defined by

$$\left\{\underbrace{A, \dots, A}_{n_1 \text{ times}}, \underbrace{\widehat{A}_1, \dots, \widehat{A}_1}_{m_1 \text{ times}}, \underbrace{A, \dots, A}_{n_2 \text{ times}}, \underbrace{\widehat{A}_2, \dots, \widehat{A}_2}_{m_2 \text{ times}}, \underbrace{A, \dots, A}_{n_3 \text{ times}}, \underbrace{\widehat{A}_3, \dots, \widehat{A}_3}_{m_3 \text{ times}}, \dots\right\}$$

and the sequence $\{B_k\}_k$ of closed convex sets in $\mathcal{H}$ defined by the same process:

$$\left\{\underbrace{B, \dots, B}_{n_1 \text{ times}}, \underbrace{\widehat{B}_1, \dots, \widehat{B}_1}_{m_1 \text{ times}}, \underbrace{B, \dots, B}_{n_2 \text{ times}}, \underbrace{\widehat{B}_2, \dots, \widehat{B}_2}_{m_2 \text{ times}}, \underbrace{B, \dots, B}_{n_3 \text{ times}}, \underbrace{\widehat{B}_3, \dots, \widehat{B}_3}_{m_3 \text{ times}}, \dots\right\}$$

It follows from (iii) and Fact 2.3 that the sequences $\{\widehat{A}_h\}_h$ and $\{\widehat{B}_h\}_h$ Attouch-Wets converge to the sets A and B respectively. Therefore, so do the sequences $\{A_k\}_k$ and $\{B_k\}_k$ (in which finite segments of constant sequences $A_k = A$ and $B_k = B$ respectively, are interposed). Consequently, considering the perturbed alternating projection sequences $\{a_h\}$ and $\{b_h\}$, with respect to $\{A_h\}_h$ and $\{B_h\}_h$ and with starting point q_0, we infer from (iv) that:

$$a_{k_N} = q_{N+1} \qquad \text{where} \quad k_N = \sum_{h=1}^{N} (n_h + m_h).$$

Since $k_N \to \infty$ as $N \to \infty$ and $\operatorname{dist}(q_{N+1}, E) \geqslant \varepsilon_0$, we deduce that the pair (A, B) is not d-stable. The proof is complete. □

We shall now establish the main result of our paper in the case where $A \cap B$ is nonempty. For the proof, we need to recall the notion of algebraic interior of a set.

Definition 3.5 Let $C \subseteq \mathcal{H}$ be a nonempty set. The *algebraic interior* of C, denoted by $\operatorname{alg\,int} C$, is the set of all points $c \in C$ such that for every $u \in \mathbb{S}_{\mathcal{H}}$ there exists $\varepsilon_u > 0$ such that $[c, c + \varepsilon_u u) \subseteq C$.

We recall that a set $C \subseteq \mathcal{H}$ is called *absorbing* if for every $x \in \mathcal{H}$, there exists a positive number α such that $x \in tC$ whenever $t > \alpha$. It is easy to see that $x \in \operatorname{alg\,int} C$ if and only if $C - x$ is absorbing. Moreover, if C is a convex set such that $\operatorname{int} C \neq \emptyset$ then, $\operatorname{alg\,int} C = \operatorname{int} C$.

We also need the following well-known corollary of Baire category theorem.

Fact 3.6 (see, e.g., [1, Corollary 3.28]) *If a complete metric space is a countable union of closed sets, then at least one of them has a nonempty interior.*

Theorem 3.7 (main result: case $A \cap B \neq \emptyset$) *Let A, B be closed convex subsets of $\mathcal{H}$ such that $A \cap B$ is nonempty. Suppose that the pair (A, B) is not regular. Then the pair (A, B) is not d-stable.*

Proof By assumption, we have $E = F = A \cap B \neq \emptyset$ and $v = 0$ (displacement vector). As in the proof of Theorem 3.4, without loss of generality we may assume that the alternating projection sequence relative to the sets A and B satisfies (3.6) for any starting point $x \in \mathcal{H}$. We can also assume that $0 \in A \cap B$.

Since the pair (A, B) is not regular, there exist $\varepsilon_0 \in (0, 1)$ and sequences $\{\delta_n\}_{n\geq 1} \subset (0, \varepsilon_0/6)$ with $\delta_n \to 0$ and $\{w_n\}_n \subset \mathcal{H}$ such that:

$$\operatorname{dist}(w_n, A \cap B) \geqslant 3\varepsilon_0 \qquad \delta_n := \max\{\operatorname{dist}(w_n, A), \operatorname{dist}(w_n, B)\}.$$

Take an arbitrary point $q_0 \in \mathcal{H}$. Let $n_1 \in \mathbb{N}$ be such that

$$\operatorname{dist}\left(\Pi_{n_1}^{A,B}(q_0),\ A \cap B\right) < \varepsilon_0.$$

Let $p_1 \in A \cap B$ be such that

$$\widetilde{p}_1 = \Pi_{n_1}^{A,B}(q_0) \in (p_1 + \varepsilon_0 \mathbb{B}_{\mathcal{H}}) \cap A$$

and let $\widetilde{a}_1 \in A$ and $\widetilde{b}_1 \in B$ be such that $\|\widetilde{a}_1 - w_1\| \leqslant \delta_1$ and $\|\widetilde{b}_1 - w_1\| \leqslant \delta_1$. Choosing $r_1 > 0$ such that $\max\{\|p_1\|, \|\widetilde{a}_1\|, 1\} \leqslant r_1$, let us define the sets

$$A_1 = A \cap r_1 \mathbb{B}_{\mathcal{H}} \qquad \text{and} \qquad B_1 = B.$$

Claim. $0 \notin \operatorname{alg\,int}(A_1 - B_1)$.

Proof of the claim. Since the pair (A, B) is not regular, by [4, Corollary 4.5], we have $0 \notin \operatorname{int}(A - B)$. Therefore, since $A_1 - B_1 \subset A - B$, we obtain $0 \notin \operatorname{int}(A_1 - B_1)$. Notice further that since A_1 is w-compact, the set $A_1 - B_1$ is closed and convex. Let us suppose, towards a contradiction, that $0 \in \operatorname{alg\,int}(A_1 - B_1)$. Then the set $A_1 - B_1$ would be an absorbing set and consequently $\cup_{n=1}^{\infty} n(A_1 - B_1) = \mathcal{H}$. In this setting, Fact 3.6 would yield that $\operatorname{int}(A_1 - B_1) \neq \emptyset$. Since $(A_1 - B_1)$ is convex, we would have $0 \in \operatorname{alg\,int}(A_1 - B_1) = \operatorname{int}(A_1 - B_1)$, a contradiction. Therefore, the assertion of the claim holds.

It follows that there exists $u_1 \in \mathbb{S}_{\mathcal{H}}$ such that, for every $\theta > 0$, we have $A_1 \cap (B_1 + \theta u_1) = \emptyset$. In particular, the closed convex sets A_1 and $(B_1 + \delta_1 u_1)$ are disjoint and A_1 is w-compact. By the Hahn-Banach theorem there exists $f_1 \in \mathbb{S}_{\mathcal{H}^*}$ and $\beta_1 \in \mathbb{R}$ such that

$$\sup f_1(B_1 + \delta_1 u_1) \leqslant \beta_1 \leqslant \inf f_1(A_1).$$

Since $p_1 \in A_1$ and $p_1 + \delta_1 u_1 \in (B_1 + \delta_1 u_1)$, there exists $s_1 \in [p_1, p_1 + \delta_1 u_1]$ such that $f_1(s_1) = \beta_1$. It follows that

$$(s_1 + \delta_1 \mathbb{B}_{\mathcal{H}}) \cap A_1 \neq \emptyset, \qquad (s_1 + \delta_1 \mathbb{B}_{\mathcal{H}}) \cap (B_1 + \delta_1 u_1) \neq \emptyset.$$

Similarly, since $\widetilde{a}_1 \in A_1$ and $\widetilde{b}_1 + \delta_1 u_1 \in (B_1 + \delta_1 u_1)$, there exists $t_1 \in [\widetilde{a}_1, \widetilde{b}_1 + \delta_1 u_1]$ such that $f_1(t_1) = \beta_1$. Since $\|\widetilde{b}_1 + \delta_1 u_1 - w_1\| \leqslant 2\delta_1$ and $\|\widetilde{a}_1 - w_1\| \leqslant \delta_1$, by convexity we have $\|w_1 - t_1\| \leqslant 2\delta_1$. Recalling that $\mathrm{dist}(w_1, A_1) < \delta_1$ we obtain

$$(t_1 + 3\delta_1 \mathbb{B}_{\mathcal{H}}) \cap A_1 \neq \emptyset \qquad \text{and} \qquad (t_1 + 3\delta_1 \mathbb{B}_{\mathcal{H}}) \cap (B_1 + \delta_1 u_1) \neq \emptyset.$$

Applying Lemma 3.2 to the sets A_1, $B_1 + \delta_1 u_1$, for

$$\delta := \delta_1, \quad r := r_1, \quad \gamma := 0, \quad \beta := \beta_1,$$

we obtain $m_1 \in \mathbb{N} \cup \{0\}$ and two (perturbed) closed convex sets $\widehat{A}_1$, $\widehat{B}_1$ such that

$$\mathrm{D_H}\left(A_1, \widehat{A}_1\right) \leqslant 9\delta_1 \quad \text{and} \quad \mathrm{D_H}\left(B_1 + \delta_1 u_1, \widehat{B}_1\right) \leqslant 9\delta_1,$$

and such that

$$\|\Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(s_1) - t_1\| \leqslant 3\delta_1.$$

Notice that

$$\mathrm{D_H}\left(A \cap r_1 \mathbb{B}_{\mathcal{H}}, \widehat{A}_1\right) \leqslant 9\delta_1 \quad \text{and} \quad \mathrm{D_H}\left(B, \widehat{B}_1\right) \leqslant 10\delta_1.$$

Since

$$\mathrm{dist}\,(t_1, A \cap B) \geqslant \mathrm{dist}\,(w_1, A \cap B) - \|w_1 - t_1\| \geqslant 3\varepsilon_0 - 2\delta_1 > 2\varepsilon_0 + 4\delta_1,$$

we have

$$\begin{aligned} \mathrm{dist}\left(\Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(s_1), A \cap B\right) &\geqslant \mathrm{dist}\,(t_1, A \cap B) - \|\Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(s_1) - t_1\| \\ &> 2\varepsilon_0 + 4\delta_1 - 3\delta_1 = 2\varepsilon_0 + \delta_1. \end{aligned}$$

Since $\|s_1 - \widetilde{p}_1\| \leqslant \varepsilon_0 + \delta_1$, we have $\|\Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(\widetilde{p}_1) - \Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(s_1)\| \leqslant \varepsilon_0 + \delta_1$ and hence

$$\mathrm{dist}\left(\Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(\widetilde{p}_1), A \cap B\right) \geqslant \mathrm{dist}\left(\Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(s_1), A \cap B\right) - (\varepsilon_0 + \delta_1) \geqslant \varepsilon_0.$$

Let us finally set

$$q_1 = \Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}(\widetilde{p}_1) = \Pi_{m_1}^{\widehat{A}_1, \widehat{B}_1}\left(\Pi_{n_1}^{A,B}(q_0)\right).$$

Iterating the same argument as above, we can inductively obtain a sequence of sets $\widehat{A}_h, \widehat{B}_h \subset \mathcal{H}$, together with points $q_h, \widetilde{p}_h \in \mathcal{H}$, $p_h \in A \cap B$, $\widetilde{a}_h \in A$, $\widetilde{b}_h \in B$, positive numbers r_h and positive integers n_h, m_h such that for every $h \in \mathbb{N}$, we have:

(i) $\widetilde{p}_h = \Pi_{n_h}^{A,B}(q_{h-1})$ and $\|\widetilde{p}_h - p_h\| < \varepsilon_0$;

(ii) $\|\widetilde{a}_h - w_h\| \leqslant \delta_h$, $\|\widetilde{b}_h - w_h\| \leqslant \delta_h$ and $\max\{\|p_h\|, \|\widetilde{a}_h\|, h\} \leqslant r_h$,

(iii) $\mathrm{D_H}\left(A \cap r_h \mathbb{B}_{\mathcal{H}}, \widehat{A}_h\right) \leqslant 9\delta_h$ and $\mathrm{D_H}\left(B, \widehat{B}_h\right) \leqslant 10\delta_h$;

(iv) $q_h = \Pi_{m_h}^{\widehat{A}_h, \widehat{B}_h}(\widetilde{p}_h) = \Pi_{m_h}^{\widehat{A}_h, \widehat{B}_h}\left(\Pi_{n_h}^{A,B}(q_{h-1})\right)$ and $\mathrm{dist}(q_h, A \cap B) \geqslant \varepsilon_0$.

Similarly to the last part of the proof of Theorem 3.4, we generate two sequences $\{A_n\}$ and $\{B_n\}$ of closed convex sets of the form

$$\{A_n\}_{n=1}^{\infty} := \left\{ \underbrace{A, \ldots, A}_{n_1 \text{ times}}, \underbrace{\widehat{A}_1, \ldots, \widehat{A}_1}_{m_1 \text{ times}}, \underbrace{A, \ldots, A}_{n_2 \text{ times}}, \underbrace{\widehat{A}_2, \ldots, \widehat{A}_2}_{m_2 \text{ times}}, \underbrace{A, \ldots, A}_{n_3 \text{ times}}, \underbrace{\widehat{A}_3, \ldots, \widehat{A}_3}_{m_3 \text{ times}}, \ldots \right\}$$

$$\{B_n\}_{n=1}^{\infty} := \left\{ \underbrace{B, \ldots, B}_{n_1 \text{ times}}, \underbrace{\widehat{B}_1, \ldots, \widehat{B}_1}_{m_1 \text{ times}}, \underbrace{B, \ldots, B}_{n_2 \text{ times}}, \underbrace{\widehat{B}_2, \ldots, \widehat{B}_2}_{m_2 \text{ times}}, \underbrace{B, \ldots, B}_{n_3 \text{ times}}, \underbrace{\widehat{B}_3, \ldots, \widehat{B}_3}_{m_3 \text{ times}}, \ldots \right\}$$

respectively, for which the variational alternating projection sequences starting from q_0 and successively projecting onto these sets do not satisfy the conditions in Definition 1.1. This shows that the pair (A, B) is not d-stable. $\square$

The above result provides a complete proof of Theorem A and positively answers the open problem considered in Problem 1.2.

4 Open problems and remarks

A typical example of convex feasibility problem where regularity fails is the so called Moment Problem. This problem consists of a pair (A, B) in a Hilbert space $\mathcal{H}$ given by a linear subspace A of finite codimension k and a Hilbert lattice cone B. This problem has an intrinsic applied nature, as the particular case of B being the cone of positive elements corresponds to the research of positive zeros of a linear functional operator via von Neumann's projection algorithm.

Bauschke and Borwein [4] proved that, for the case of codimension $k = 1$, the alternating projection sequences always converge to a point in $A \cap B$; at the same time, they provided an example (see [4, Example 5.5]) where the pair (A, B) has unbounded intersection and is not regular, and so is not d-stable thanks to Theorem A. In particular, this means that there exists a sequence of Attouch-Wets approximations for A and B for which the corresponding variational projection sequences fails to converge: however, as we have seen, an explicit construction of such sequences is a difficult problem.

A posteriori, the fact that d-stability does not hold for these instances of Moment Problem is not surprising since the involved lattice cone B has usually an unbounded basis (and the approximation of such cones is often problematic). For the time being, it remains unknown if the alternating projection sequences converges in norm for a general moment problem of codimension $k > 1$. Our results suggest that the variational approach does not seem to be the proper way to investigate the Moment Problem and different techniques related to the lattice structure of the cone are required, especially for higher codimension k, see [3].

To summarize, we hereby established the equivalence between regularity and d-stability for a pair of closed convex subsets (A, B) of $\mathcal{H}$ provided that $A \cap B$ is nonempty and bounded (or more generally, if the best approximation sets E and F are nonempty and bounded). Example 5.2 in [11] shows that regularity does not imply

d-stability of the pair (A, B) when $A \cap B$ is unbounded, even in a finite-dimensional setting.

At this point, let us recall from [8, Theorem 3.7] that the alternating projection sequence $a_n = P_A P_B(a_{n-1})$, $n \geqslant 1$, is *self-contracted*, that is, for any $N \geqslant 1$ the finite sequence $\{\|a_i - a_N\|\}_{1 \leqslant i \leqslant N}$ is decreasing. This fact guaranties that the set of accumulation points of the alternate projection sequence $\{a_n\}_{n \geqslant 1}$ is at most a singleton (in particular, the method converges in finite dimensions, whenever the best approximation sets are nonempty, defining, in addition, a polygonal curve of finite length, see [10] e.g.). The property of self-contractedness is completely lost in case of inexact projections or in case of variational perturbations of the sets, changing the nature of the problem. Therefore, an interesting question is to determine whether there exists a weaker regularity condition on (A, B) that is equivalent to the norm convergence of the alternating projection method.

Another possible direction for further research concerns regularity and d-stability for more than two closed convex sets or for different iterative projection algorithms (see [2, 9] e.g.). Indeed, the convex feasibility problem for more than two sets is a rich field of research, where the order (and the frequency) on which the alternating projections are taken would potentially lead to different conclusions and unexplored notions of regularity.

Acknowledgements Major part of this research has been done during a research visit of the second author to the Università Cattolica del Sacro Cuore (Milano, June 2025). This author thanks his hosts for hospitality. The second author thanks Stoyan Apostolov and Eva Kopecká for useful discussions. The authors thank anonymous referees whose comments and remarks considerably improved the presentation of this work. The first author was partially supported by INdAM-GNAMPA. The research of the second author was supported by the INdAM (GNAMPA, Professore Visitatore) and the Austrian Science Fund (Grant FWF 10.55776/P36344). The research of the third and fourth authors was supported by the INdAM -GNAMPA Project, CUP E53C23001670001, and by the MICINN project PID2020–112491GB-I00 (Spain). For open access purposes, the second author has applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

Funding Open access funding provided by TU Wien (TUW).

Declarations

Conflict of interest The authors declare that they have no relevant financial or non-financial conflict of interest to disclose.

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